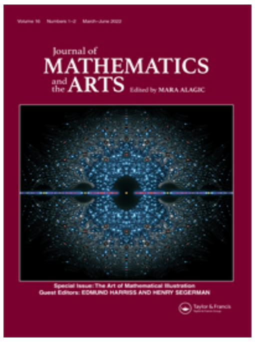

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Quasimusic: tilings and metre

Rodrigo Treviño

Department of Mathematics, University of Maryland, College Park, MD, USA

ABSTRACT
In this paper, I try to explain how, by using concepts and ideas from the mathematical theory of tilings, we can approach metre in music through a geometric and algebraic point of view, being pinned down by a subgroup of $\mathbb{R}$ with the hierarchical structure, leading to an abstract approach to rhythm, tempo and time signatures. I will also describe an algorithmic approach to write down sound using this structure which gives a way in which music can be written in an irrational metre.

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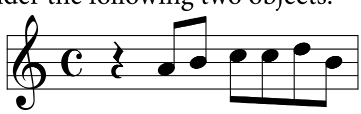
Tilings; music; rhythm

In this paper, I try to use concepts from the theory of aperiodic tilings to generalize the concept of rhythm, tempo, and time signatures. This helps both illustrate the concepts related to tilings as well as make the notions of musical metre more abstract, allowing one to approach musical composition from a more general point of view. Let me begin by suggesting how tilings and music can be related, or how they can be seen to have similar underlying structures. Consider the following two objects.



The first object is a bar of music containing part of a melody while the second can be thought of as a finite collection of tiles which tile a line segment. The relative placement of the two images is deliberate: the fact that the coloured tiles cover a one-dimensional segment in the bottom image suggests that the bar of music is tiled by the notes and rests found in the bar. The distinction between notes of same length and different pitch is reflected in the difference in colour of tiles of the same length in the image on the bottom.

Western music is traditionally organized through an ordered and hierarchical structure.[1] It is hierarchical since we make notes fit into bars which fit into movements, etc. It is ordered because time flows in one direction and music is written in this order-respecting way, meant to indicate the order in which notes are played. That music in the Western tradition can be seen as the tiling of time with sounds is the basic idea that motivates this paper. Here I try to show how the mathematics of aperiodic tilings, that is, the

CONTACT Rodrigo Treviño rodrigo@trevino.cat

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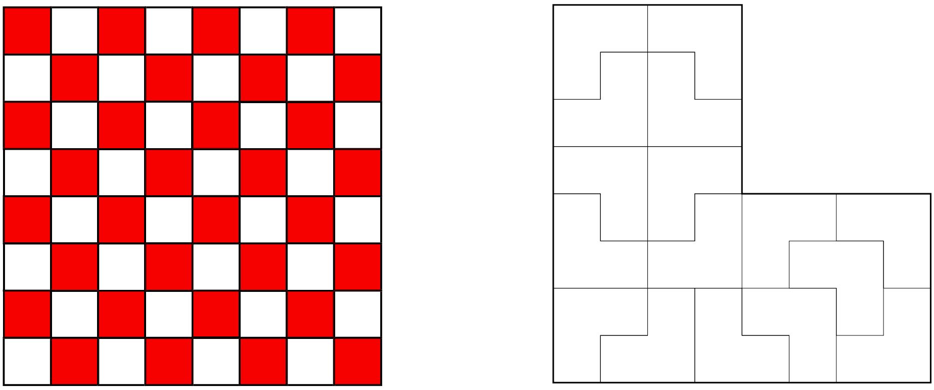

Figure 1. A checkerboard tilings (left) and *a* chair tiling (right).

language used to describe the hierarchical and geometric structure of tilings, can be useful in describing the metre structure of Western music.

The main mathematical objects here are tilings, especially ones which are aperiodic. Aperiodicity will be defined in Section 1, and is heavily contrasted with periodicity (see Figure 1 for examples of periodic and aperiodic tilings). The study of aperiodic tilings lies at the intersection of several areas of mathematics and physics.[2] Kepler explores the problem of finding a tiling of the plane with finitely many shapes derived from the pentagon in his *Harmonices Mundi* from 1619. The problem of finding an aperiodic tiling of a plane was not formalized until the work of Hao Wang in the early 1960s, where he asked whether it was decidable that a specific set of tiles (known as *Wang tiles*) could tile the plane. Through the use of Turing machines, his student Robert Berger answered this question in the negative, which shows that there are sets of Wang tiles that can tile the plane aperiodically. In the 1970's, Roger Penrose found two shapes derived from a pentagon, which can tile the plane only aperiodically and with a five-fold symmetry, giving the well-known Penrose tiling.

The connection with physics came a few years later: Dan Shechtman studied (Shechtman et al., 1984) a solid whose atomic structure seemed impossible up to that point since solids with aperiodic structure were not known. These solids are now known as quasicrystals. For this discovery, he received the Nobel Prize in Chemistry in 2011. As such, aperiodic tilings serve as good models for quasicrystals.

The concept of tilings entered the musical realm with the work of Vuza (1991, 1992a, 1992b, 1993), although this had already been hinted at by the composer Olivier Messiaen. Some of his concepts were developed into what became known as *tiling canons*, which have been the subject of many works, even having special issues of the *Journal of Mathematics and Music* and *Perspectives of New Music* each solely dedicated to tilings in music (see also Hall and Klingsberg, 2006). See Andreatta and Agon (2009) and Rahn (2011), respectively, for an overview of the collections of papers in those special issues. Most of these works are concerned with periodic tilings.

One-dimensional tilings, the ones which are relevant in music, are tied to sequences of symbols. In parallel to the use of tilings, the use of sequences has also been explored musically, especially for aperiodic sequences: Canright (1990) explores rhythm patterns associated with the Fibonacci sequence. In Carey and Clampitt (1996), Carey and Clampitt explore pitch structures derived from sequences with particular focus in sequences with self-similar structures, that is, derived from substitution rules, where it is briefly mentioned how rhythm structures can be assigned from sequences and that these come ordered hierarchically. This line of thought is picked up by Callender in Callender (2013, 2015), where the focus is on Sturmian sequences[3] and interpreting the relationship between different hierarchical levels as a canon. The use of substitution systems and aperiodic sequences also show up in algorithmic composition using so-called Lindenmeyer systems (see e.g. Manousakis, 2006; Mason and Saffle, 1994; Prusinkiewicz, 1986; Worth and Stepney, 2005). The recent article of Ong (2020) also considers the use of aperiodic functions in musical composition. The contemporary British composer Liam Taylor-West, after spending a year as the resident composer of the School of Mathematics at the University of Bristol, has recently finished a series of works derived and inspired by substitution tilings. There is an applet in Greg Egan's webpage to create music based on two-dimensional aperiodic tilings, which is only superficially related to what is discussed here. Likewise, there is an approach on planar tilings by Skala (n.d.).

In the spirit of the special issue, here I intend to use some of the mathematics used in the study of aperiodic tilings to push the use of tilings in music further. This will help both illustrate the mathematics of aperiodic tilings as well as bring new ideas to the theory of rhythm. What sets the point of view here apart from previous aperiodic approaches is that I do not only focus on combinatorial aperiodicity, but also on geometric aperiodicity. This roughly means that not only will we be interested in aperiodic sequences and deriving sounds from them, but we are interested in the *length* of the sounds attached so each symbol in an aperiodic sequence. This length, a type of beat, is tied to the geometry of the tiling in this analogy and is related to certain algebraic properties of the tiling. Most (though not all, the notable exception being Callender, 2013) previous work using aperiodic structures in music remains committed to the use of a rational beat even when it may be more natural to do otherwise. Here I suggest how we can get out of this rational trap by giving metre an algebraic structure. This leads to a sturdy notion of time signatures, tempo and rhythm using the language of tilings and semigroups, and helps define irrational metres (see Section 2).

I confess that the fact that one can have a truly irrational metre, as I argue here, does not mean that there is a reasonable way to write it down for someone to play. In other words, even if there is good notation for it, it is most likely impossible to be played by humans. Performing irrational metre is no mere abstract concern: see Callender (2014) for a thorough look at the mathematical, compositional, and performance issues involved in the approximation of irrational rhythms by live performers, focusing on those found in the work of Nancarrow.

Since there are several threads in this mathemusical braid, I will take the time to introduce each of them and later describe how they come together. I will make an attempt to minimize how technical the exposition is. However, I will need to introduce enough technical language to be able to discuss the main ideas and I will illustrate the ideas with examples to make the ideas more concrete. By the end of this article, however, I hope to convince you that there *is* such a thing as an irrational time signature, and how you can compose in $\sqrt{5}$. In a follow-up paper (Treviño, 2021), I will develop the algebraic theory of metre separately without appealing to aperiodic tilings.

This paper is organized as follows. In Section 1, all the necessary notions from tilings will be introduced and illustrated with examples. In Section 2, I show how musical metre is defined from a one-dimensional tiling. Section 3 discusses how one can put together the ideas of Sections 1 and 2 to make music in the case of self-similar tilings. Section 4 covers several composition tricks which come from the algebraic structure of metre, again focusing on the self-similar case. The focus here is on algorithmic composition, with examples illustrating the ideas presented, all of which were created using a computer. It would be interesting to try to compose non-algorithmically in some irrational metre.

Note: the electronic version of this document has links to webpages and audio files online.

1. Tilings

Tilings are ways of covering a Euclidean space using compact, connected objects called tiles. A tiling of some Euclidean space will be denoted by the symbol $\mathcal{T}$. I will give a very soft introduction to some concepts related to their study. The interested reader looking for in-depth treatment can consult any of the excellent introductions to the topic: Grünbaum and Shephard (2016); Arthur Robinson (2004) and Baake and Grimm (2013); Sadun (2008). There is also an excellent online resource with numerous examples, the Tilings Encyclopedia.

1.1. Warmup: hierarchical tilings in two dimensions

The easiest way to visualize tilings is in two dimensions. Figure 1 has some local tiling designs for the floor, walls or ceiling in a room you may visit. It is easy to imagine tiling an infinite plane using the checkerboard pattern by extrapolating the periodic pattern indefinitely. What is harder to imagine is how to find a good rule to extrapolate the second image to a tiling of an infinite plane. Here is the good news: not only can the second pattern be extended indefinitely, but it can be done so that the resulting tiling, unlike the infinite checkerboard tiling, will have an aperiodic structure. By this, I mean the following: there is no vector $\mathbf{v}$ such that if I translate the infinite tiling by this vector then the resulting tiling will be exactly the same as the one I started with. The checkerboard tiling obviously has such a vector. In fact, it has infinitely many vectors with this property.

There is a very satisfying way to obtain tilings of the plane which are aperiodic, and this is by using a substitution and inflation rule. First, let's agree on some terminology: tilings of the plane are made up of tiles which only meet at their boundaries. I will also assume here that for each tiling we consider there are only finitely many tile types. That is, there is a finite collection of tiles $\{t_1, \ldots, t_k\}$, called the prototiles, such that any tile on the tiling is an unrotated copy of one of the t_i. The set of prototiles for the checkerboard tiling in Figure 1 consists of two elements, the white tile and the red tile. The set of prototoles for the chair tiling[4] in Figure 1 has 4.

An inflation and substitution rule on a set of prototiles is the operation of inflating each prototile by the same factor $\lambda > 1$ and then tiling the bigger tiles with copies of the original prototiles. Figure 2 denotes the inflation and substitution rules used to generate the tilings in Figure 1.

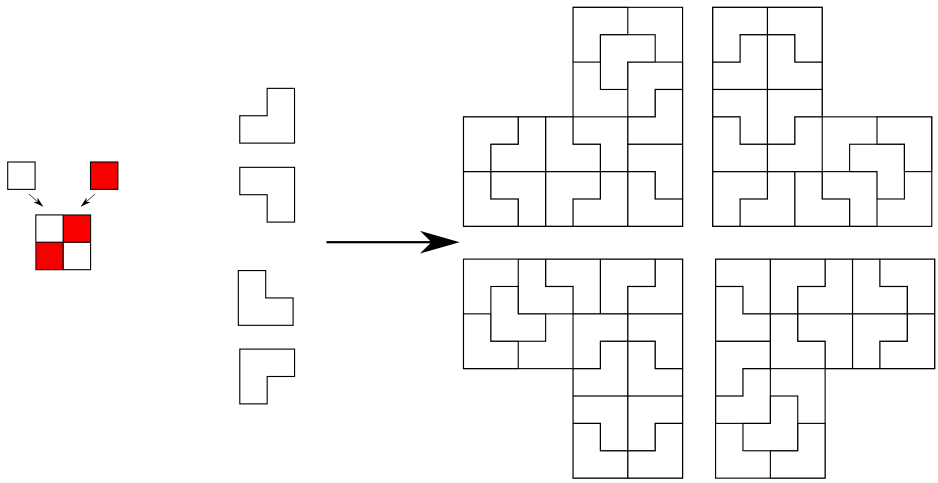

Figure 2. The inflation and substitution rule behind the checkerboard tiling (left), and the one behind the chair tiling (right). The first one has inflation factor $\lambda = 2$ while the other one has $\lambda = 4$.

The good news is that by having an inflation and substitution rule we can get a tiling of the plane. The idea is roughly the following: pick a prototile and apply the inflation and substitution rule to obtain a finite collection of tiles which tile an enlarged copy of the starting prototile which we call a patch. Now if we apply again the inflation and substitution rule to each tile in the patch we obtain an even larger copy of the original prototile which is tiled by copies of the prototiles. Doing this procedure arbitrarily many times, we obtain arbitrarily large patches which are tiled by copies of prototiles.[5] At this point, one has to be somewhat careful in taking limits. I will not go into details of how this is done but rest assured there are ways of taking limits so that the limiting object will be a tiling of the plane. The resulting tiling may or may not be aperiodic. Indeed, the two examples in Figure 2 show an inflation and substitution rule which lead to tilings of the plane but one is aperiodic and the other one is not (this will be discussed in Section 1.3).

In the study of tilings, it is very convenient to see them as objects organized through a hierarchical structure. Roughly speaking, a hierarchical structure is a way of organizing finite sets of tiles into bigger collections of tiles, called supertiles and organizing supertiles into even larger supertiles and so on. Instead of formally defining them, it is easier to see them through examples.

Consider a tiling constructed through an infinite application of an inflation and substitution rule as described above. The first step was to apply the inflation and substitution rule to a prototile to obtain a rescaled copy of the prototile which is tiled by copies of prototiles. If we do this to every prototile, we obtain all possible level-1 supertiles (see Figure 2 again). Applying the inflation and substitution rule to any level-1 supertiles we obtain level-2 supertiles, which are tiled by level-1 supertiles the way level-1 supertiles are tiled by prototiles. Applying the inflation and substitution rule n times, we obtain level-n supertiles. By construction, level-$(n-1)$ supertiles tile level-n supertiles the same way prototiles tile level-1 supertiles. As such, it is natural to consider prototiles as level-0 supertiles.

The structural hierarchy of a tiling then refers to the way level-$(n-1)$ supertiles come together to form level-n supertiles for all $n \geq 0$. Figure 3 shows the first three hierarchical

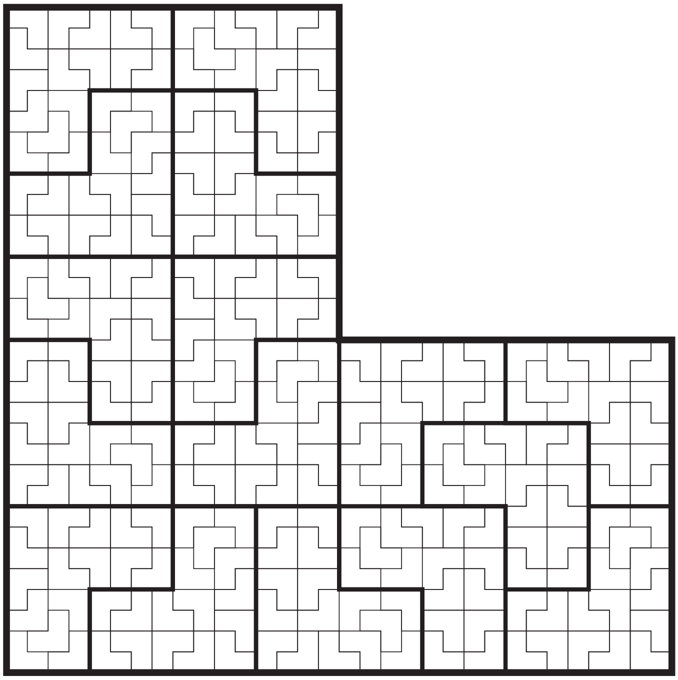

Figure 3. A level-2 supertile tiled by level-1 supertiles which are tiled by level-0 supertiles. Level-1 supertiles are tiled according to the rule in Figure 3.

levels for the chair tiling in Figures 1 and 2. For tilings constructed by the application of an inflation and substitution rule as above, the hierarchical structure is independent of the level of hierarchy since, for any $n > 1$, level-$(n-1)$ supertiles tile level-n supertiles the same way prototiles tile level-1 supertiles. Tilings which have this property are called self-similar. Thus, tilings constructed from an inflation and substitution rule are self-similar tilings.

1.2. One-dimensional tilings

Now that we have introduced and developed enough language to discuss tilings, we want to focus on tilings in one dimension, that is, tilings of the real number line. All the motivating examples above were in two dimensions, but one-dimensional tilings are somewhat easier to treat for two reasons:

(i) All tiles in a tiling are intervals. Unlike tiles in higher dimensions, the shape of tiles is all the same. We can distinguish two tiles in a tiling by the length of the tile and, if applicable, also by label. The same applies to supertiles.
(ii) Tiles in a one-dimensional tiling are ordered. That is, if we have two tiles t_1 and t_2 then either t_1 comes before t_2, t_2 comes before t_1, or $t_1 = t_2$. One cannot easily make this type of statement for tilings in higher dimensions.

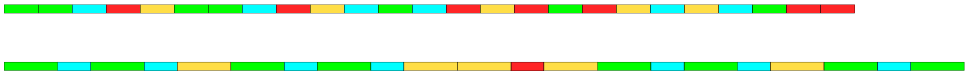

Figure 4. Two patches corresponding to different one-dimensional tilings. Although the colours are the same, these two examples are not meant to be related in any way as they were constructed using different rules.

In order to talk about one-dimensional tilings, it suffices the describe the location of every tile in a tiling. Since all tiles are intervals, we adopt the convention that we will use the left endpoint of a tile to describe its location and, if needed, its type. Before this gets too abstract, consider the one-dimensional tilings in Figure 4.

The first example in Figure 4, contains a piece of a tiling where all tiles have length 1, but with 4 different prototiles which are distinguished by colours. The second example contains a piece of a tiling also with 4 prototiles, ordered as {green, red, yellow, blue}, but they each have respective length (proportional to) $\{\tau, 1, \tau, 1\}$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Without loss of generality, we can assume that any tiling has a tile whose left endpoint sits at 0 in the number line, and so the entire tiling can be described by an infinite string of symbols, e.g.

$$\cdots ADCDDABC.CBADDCDABAABC \cdots ,$$

where the dot . was placed in the corresponding spot where two tiles surround 0 in the number line. This type of symbolic description is possible for any one-dimensional tiling. However, when the tiles do not all have the same length, some information is lost. Therefore, to describe a one-dimensional tiling one needs:

(i) (without loss of generality) to assume the left endpoint of a tile coincides with 0 in the real number line,
(ii) a bi-infinite string of symbols representing the different tile types in the order in which they appear in the tiling, with the location of a dot . in the string representing which tiles surround the origin,
(iii) and a bi-infinite collection of increasing points $\Lambda = \{\ldots, p_{-2}, p_{-1}, p_0 = 0, p_1, p_2, \ldots\}$ representing the location of the left endpoint of every tile.

In any example where all tiles have the same length, we can take $\Lambda = \ell \cdot \mathbb{Z}$, where ℓ is the length of the tiles.

1.2.1. The group of tile lengths

Suppose that a one-dimensional tiling $\mathcal{T}$ has finitely many prototiles $\{t_1, \ldots, t_k\}$ with lengths $\{\ell_1, \ldots, \ell_k\}$. The group of tile lengths is formally the set of all integer linear combinations of the lengths of the prototiles:

$$\Gamma_{\mathcal{T}} := \left\{ \tau = \sum_{i=1}^{k} n_i \ell_i : (n_1, \ldots, n_k) \in \mathbb{Z}^k \right\},$$

which is the group generated by differences of the semigroup of tile lengths:

$$\Gamma_{\mathcal{T}}^{+} := \left\{ \tau = \sum_{i=1}^{k} n_i \ell_i : n_i \in \mathbb{Z},\ n_i \geq 0 \right\},$$

that is, $\Gamma_{\mathcal{T}} = \Gamma_{\mathcal{T}}^{+} - \Gamma_{\mathcal{T}}^{+}$. The hierarchical structure of one-dimensional tilings implies that the length of a level-n supertile is an element of $\Gamma_{\mathcal{T}}$ for all n. The set $\Gamma_{\mathcal{T}}$ is the free abelian group of rank k with generators $\{\ell_1, \ldots, \ell_k\}$, and so we can add and subtract elements of $\Gamma_{\mathcal{T}}$. In the first example in Figure 4, all the tiles have the same length, and so the group is $\Gamma_{\mathcal{T}} = \ell \cdot \mathbb{Z}$, where ℓ is the length of any of the prototiles.

1.2.2. Self-similar one-dimensional tilings

Constructing self-similar tilings in one dimension is simple. As in any dimension, it can be done through an inflation and substitution rule. However, because the geometry of the tiles in one dimension is much simpler than in higher dimensions, it suffices to describe the substitution rule. The inflation factor can be extracted from it.

Let us see this through examples. As described above, to each one-dimensional tiling made up of finitely many prototiles one can assign a bi-infinite string of symbols which describes the order of the tiles. So we can start by describing a symbolic substitution rule, for example, these two:

$$A \mapsto AADBC, \quad B \mapsto BABDC, \quad C \mapsto CDABB, \quad D \mapsto DADBC, \tag{1}$$

or

$$A \mapsto ADA, \quad B \mapsto BA, \quad C \mapsto CBC, \quad D \mapsto DC. \tag{2}$$

Starting with the symbol A, iterating the symbolic substitution leads to the following sequence of words[6]:

$$A \mapsto AADBC \quad \mapsto AADBC\,AADBC\,DADBC\,BABDC\,CDABB \mapsto \cdots$$

for (1), or for (2):

$$A \mapsto ADA \quad \mapsto ADADCADA \ \mapsto\ ADADCADADCCBCADADCADA \mapsto \cdots.$$

The question is now how to turn the symbolic substitution into a geometric one. That is, how can we find the right lengths of each prototile and an inflation factor that will respect the combinatorics of the substitution rule. Let us begin with the symbolic substitution in (1). We are looking for four non-negative numbers $\ell_A, \ell_B, \ell_C, \ell_D$, which will be the lengths of each prototile, and a factor $\lambda > 1$ such that, for example, when we inflate a prototile t_A by a factor of λ we obtain a level-1 supertile of length $\lambda\ell_A$ which is tiled by 2 copies of t_A and one copy of each t_B, t_C, t_D (this is dictated by the symbolic substitution rule $A \mapsto AADBC$ in (1)). Doing this sort of analysis for each prototile, the quantities we see that need to satisfy

$$\begin{aligned} 2\ell_A + \ell_B + \ell_C + \ell_D &= \lambda \ell_A \\ \ell_A + 2\ell_B + \ell_C + \ell_D &= \lambda \ell_B \\ \ell_A + 2\ell_B + \ell_C + \ell_D &= \lambda \ell_C \\ \ell_A + \ell_B + \ell_C + 2\ell_D &= \lambda \ell_D. \end{aligned} \tag{3}$$

Letting

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \bar{\ell} = (\ell_A, \ell_B, \ell_C, \ell_D),$$

the system (3) is equivalent to the eigenvalue/eigenvector equation $M\bar{\ell} = \lambda\bar{\ell}$ for the matrix M above defined by the substitution. This matrix is called the substitution matrix*[7] The substitution matrix M of a symbolic substitution has entry M_{ij} equal to the number of tiles of type j which fit into the inflated copy of a tile of type i. It is straight forward to verify that $\ell_A = \ell_B = \ell_C = \ell_D = 1$ and $\lambda = 5$ is a solution to (3). This is an example of a constant-length substitution, that is, a substitution for which there is an $n > 1$ such that each symbol is substituted into exactly n symbols. Whenever this happens, the corresponding inflation factor is $\lambda = n$.

Doing the same for the symbolic substitution described in (2), we obtain

$$\begin{aligned} 2\ell_A + \ell_D &= \lambda \ell_A \\ \ell_A + \ell_B &= \lambda \ell_B \\ \ell_B + 2\ell_C &= \lambda \ell_C \\ \ell_C + \ell_D &= \lambda \ell_D \end{aligned} \tag{4}$$

which has solution

$$\lambda = \tau^2 = \frac{3 + \sqrt{5}}{2},$$

which is a real number approximately 2.618033989, and $(\ell_A, \ell_B, \ell_C, \ell_D) = (\tau, 1, \tau, 1)$. Assigning the colour green to tiles of type A, red to tiles of type B, yellow to tiles of type C, and blue to tiles of type D, the reader can compare the symbolic substitutions in (1) and (2) to the tilings in Figure 4.

Note that the solutions to the equations of the form $M\bar{\ell} = \lambda\bar{\ell}$ above only make geometric sense if $\lambda > 1$ and each entry in $\bar{\ell} = (\ell_A, \ell_B, \ell_C, \ell_D)$ is a positive real number. Were we just lucky in obtaining solutions that make geometric sense? Are there other solutions that make geometric sense?

Definition 1.1: A square matrix M is primitive if there is a $k \in \mathbb{N}$ such that M^k has all positive entries. A symbolic substitution rule is a primitive substitution rule if its substitution matrix is a primitive matrix.

Theorem ((Perron–Frobenius)): *Let M be a primitive matrix of non-negative integers. Then there exists a unique vector $\bar{v}_{PF}$, of length 1 and with all positive entries, and $\lambda_{PF} > 1$ such that $M\bar{v}_{PF} = \lambda_{PF}\bar{v}_{PF}$. These are called the Perron–Frobenius eigenvalue/eigenvector.*

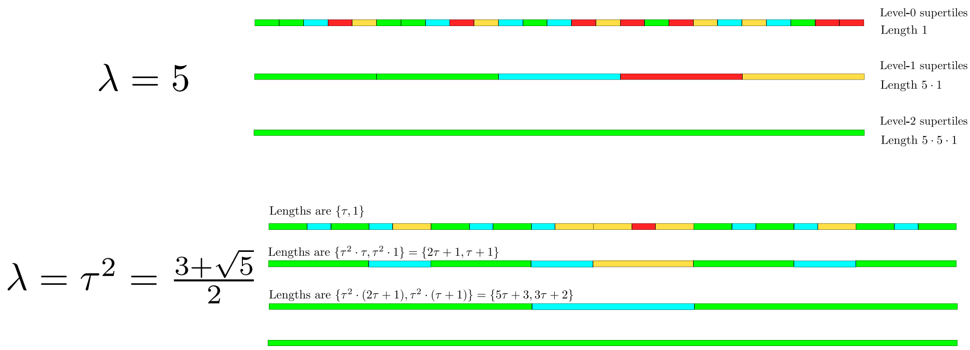

Figure 5. Self-similar hierarchical structure for the examples in this section.

Corollary 1.1: *For any primitive substitution rule, there is a unique choice of lengths (up to scaling) and inflation factor defining a self-similar one-dimensional tiling.*

In this case, the inflation factor λ will always be a real Perron number (Lind, 1987), that is, a positive algebraic integer whose Galois conjugates are less than $|\lambda|$ in modulus. As such, when M is primitive, the equation $Mv = \lambda_{PF} v$ forces the group of tile lengths $\Gamma_{\mathcal{T}}$ to be a subgroup (up to a scaling) of the finitely generated group $\mathbb{Z}[\lambda]$, the ring of integers defined by λ. Note that if the lengths of the prototiles for a primitive substitution are $\{\ell_A, \ldots, \ell_k\}$, then the lengths of different level-n supertiles are $\{\lambda^n \ell_A, \ldots, \lambda^n \ell_k\}$.
Let us revisit the examples from Figure 4. Depicted in Figure 5 are patches from the tilings obtained from the substitutions in (1) and (2) along with patches of the tilings made up of level-n supertiles for small values of n. Since the first example of Figure 4 has the same length (which without loss of generality we assume is 1) for all tiles and the inflation factor is $\lambda = 5$, every level-n supertile has length 5^n. For the second example, the tiles have a length either 1 or τ. Using the fact that τ is a solution to $x^2 - x - 1 = 0$, it follows that level-1 supertiles have length either $\lambda \cdot 1 = \tau^2 \cdot 1 = \tau + 1$ or $\lambda \cdot \tau = \tau^3 = \tau(\tau + 1) = \tau^2 + \tau = 2\tau + 1$, where the last equality follows from the fact that $\tau^2 = \tau + 1$ by definition. Both of these quantities are elements of the group $\Gamma = \mathbb{Z}[\tau] = \mathbb{Z} + \tau\mathbb{Z}$ of tile lengths.

Let me now note that it follows that the group of tile lengths of a one-dimensional, primitive, self-similar tiling depends only on the substitution matrix. So any two symbolic substitution rules with the same substitution matrix will have the same set of prototiles and thus the same group of tile lengths. As an example, we can re-arrange the symbols in the symbolic substitutions given in (1):

$$A \mapsto ADBCA, \quad B \mapsto BADCB, \quad C \mapsto CDBAB, \quad D \mapsto DABDC$$

or, for (2):

$$A \mapsto AAD, \quad B \mapsto BA, \quad C \mapsto BCC, \quad D \mapsto DC.$$

Doing so gives new symbolic substitutions but the substitution matrix remains the same. As such, the tile lengths of the corresponding tilings and the group of tile lengths do not change.

1.3. Aperiodicity versus repetitivity

Let us go back to the two-dimensional examples in Figure 1. As discussed, both of these are constructed from an inflation and substitution rule. Although one grows to be a periodic tiling while the other one grows to be aperiodic, both of them are repetitive. This means that for any patch that you may see in the tiling there is an R such that the ball of radius R around *any* point contains a translated copy of the patch. It is a theorem in symbolic dynamics that any primitive substitution rule will give a repetitive tiling $\mathcal{T}$. In other words, for any patch in $\mathcal{T}$ there is a $R > 0$ such that there is a copy of that patch found in $\mathcal{T}$ within a distance of R of any tile in $\mathcal{T}$. What I want to emphasize here is that a tiling can repeat but not repeat periodically.

2. The algebraic structure of metre

The goal of this section is to connect the tiling notions mentioned in earlier sections with the basic objects of musical metre. By doing this not only will we put the notion metre in Western music on some abstract mathematical footing, and this abstract mathematical footing will allow us make the concept of metre more general.

Let $\mathcal{T}$ be a one-dimensional tiling with finitely many prototiles and $\Gamma_{\mathcal{T}} = \Gamma^+_{\mathcal{T}} - \Gamma^+_{\mathcal{T}}$ its group of tile lengths. From such a tiling we define:

Bars/measures defined by $\mathcal{T}$ level-1 supertiles of $\mathcal{T}$;

Beats/note duration defined by $\mathcal{T}$ elements of a semigroup $\Gamma'_+ \subset \mathbb{R}^+$ with $[\Gamma'_+ - \Gamma'_+, \Gamma_{\mathcal{T}}] < \infty$, that is, Γ'_+ generates a group within which $\Gamma_{\mathcal{T}}$ has finite index. *This is a choice*, not something canonically defined from a tiling, although it is restricted by it;

Tempo defined by $\mathcal{T}$ A tiling $\mathcal{T}$ automatically defines a tempo as follows. The frequency of tiles in a given tiling $\mathcal{T}$ is quantity

$$\mathrm{freq}(\mathcal{T}) = \lim_{N\to\infty} \frac{\text{number of tiles of } \mathcal{T} \text{ in } [0,N]}{N},$$

which exists in many cases, including tilings constructed from primitive substitutions. The tempo defined by $\mathcal{T}$ is

$$\mathrm{tempo}(\mathcal{T}) = 60 \cdot \mathrm{freq}(\mathcal{T}) \frac{\text{beats}}{\text{minute}}. \tag{5}$$

If the endpoints of the tiles are $\mathbb{Z}$ (as in classical Western music), then the frequency is 1. Thus, the 60 appearing in (5) reflects the convention that time flows with unit time, in which case the tiling defined $\mathbb{Z}$ has a tempo of 60 beats per minute. In the case of $\mathbb{Z}$, then this also defines a pulse. However, for tilings with a more complicated group of tile lengths, a tempo will be defined without a pulse necessarily being defined.

Let us now examine how we can fit basic Western metre into this framework, the most basic of which is a $\frac{4}{4}$ time signature with tempo defined by ♩ $= \mathfrak{t} > 0$. Consider the symbolic substitution rule $A \mapsto AAAA$ which very evidently has expansion constant $\lambda = 4$. The resulting (periodic) tiling is, up to a scalar multiple, given by $\mathbb{Z}$. Every level-1 supertile (i.e. an interval of $4\mathbb{Z} \subset \mathbb{Z}$) is tiled by 4 tiles, and so each bar is equally divided into 4 units of equal length. The right scaling of $\mathbb{Z}$ to obtain the tempo ♩ $= \mathfrak{t}$ is by $60/\mathfrak{t}$ since

$$\operatorname{tempo}\left(\frac{60}{\mathfrak{t}}\mathbb{Z}\right) = 60 \cdot \operatorname{freq}\left(\frac{60}{\mathfrak{t}}\mathbb{Z}\right)\frac{\text{beats}}{\text{minute}} = 60 \cdot \frac{\mathfrak{t}}{60}\frac{\text{beats}}{\text{minute}} = \mathfrak{t}\frac{\text{beats}}{\text{minute}}.$$

Note that the associated group for this metre structure is $\Gamma_{\mathcal{T}} = \frac{60}{\mathfrak{t}}\mathbb{Z}$ which is generated by the semigroup $\Gamma_{\mathcal{T}}^{+} = \frac{60}{\mathfrak{t}}\mathbb{N}$.

It remains to define the beats in this metre structure. As defined above, it remains to define a group Γ' which contains $\frac{60}{\mathfrak{t}}\mathbb{Z}$ as a subgroup of finite index, and with the property that it is generated by a positive semigroup Γ'_{+} which contains the basic beats. There are several ways of doing this (in fact, infinitely many). Here is the example of one, picked to allow both triplets and beats as short as one-hundred-and-ninety-twoth notes: letting

$$\Gamma'_{+} = \frac{\lambda^{-2}}{3}\Gamma_{\mathcal{T}}^{+} = \frac{1}{4^2 3}\frac{60}{\mathfrak{t}}\mathbb{N} = \frac{5}{4\mathfrak{t}}\mathbb{N} \tag{6}$$

then this semigroup of beats divides the basic notes (which are tile lengths ♩ $= |t|$) in up to $2^4 \cdot 3 = 3 \cdot 16 = 48$ beats of equal duration. From these we can generate eighth notes (♪ $=$ ♩$/2 = |t|/2$), sixteenth notes (𝅘𝅥𝅯 $=$ ♪$/2 =$ ♩$/4 = |t|/2^2$), thirty-second notes (𝅘𝅥𝅰 $=$ ♩$/2^3 = |t|/2^3$), sixty-fourth notes (𝅘𝅥𝅱 $=$ 𝅘𝅥𝅰$/2 =$ ♩$/2^4 = |t|/2^4$), triplets (/3), and even one-hundred-and-ninety-twoth notes ($\frac{♩}{2^4 . 3}$).

Now consider a primitive self-similar one-dimensional tiling $\mathcal{T}$ with inflation factor λ, where we assume that the minimal polynomial defining λ is of degree d. In this case, we have that $\Gamma_{\mathcal{T}} \subset s \cdot \mathbb{Z}[\lambda]$ for some $s > 0$. The natural choices for the semigroup of beats are ones of the form $\Gamma'_{+} = q^{-\ell}s \cdot \mathbb{Z}_n[\lambda]$ for some $q, \ell, n \in \mathbb{N}$, where

$$\mathbb{Z}_n[\lambda] = \left\langle \lambda^{d-1}, \dots, 1, \lambda^{-1}, \dots, \lambda^{-d}, \lambda^{-d-1}, \dots, \lambda^{-2d}, \dots, \lambda^{-nd} \right\rangle,$$

that is, $\mathbb{Z}_n[\lambda]$ is the group of integer combinations of powers of λ, where the powers range from $-nd$ to $d-1$. Note that when λ is an integer then $\Gamma'_{\mathcal{T}} = q^{-\ell}s \cdot \mathbb{Z}$ for some $q \in \mathbb{N}$ which is multiple of λ. These generalize the choice of semigroup (6). As such, the defining algebraic object for a time signature is the (finitely generated) module $s \cdot \mathbb{Z}[\lambda]$ for an algebraic integer λ. In the next two sections, we will show how one can use these notions concretely in making music.

3. Quasimusic

I now want to talk about how one can take all that has been covered and produce sounds that could potentially be called music. Let me start with an example.

Consider the tilings from the top half of Figure 5, which were constructed using the substitution rule in (1). The three tilings shown in the top half of Figure 5 are: the tiling obtained by (1), the tiling by level-1 supertiles of the same tiling and the tiling by level-2 supertiles of the same tilings. These three tilings fit nicely together, that is, for each level-1 supertile, there is a collection of tiles which tile it, for each level-2 supertiles there is a collection of level-1 supertiles which tile it, etc. As such the length of each level-k supertile is a (positive) element of our group of tile lengths, and so it corresponds to a beat. As such, one can assign a note or sound to each colour of each of the tilings, and play them together. For example, assigning (using keyboard notation)

$$\text{green} = C_5,\ \text{blue} = G_5,\ \text{red} = D_5,\ \text{ and yellow } = F_5 \tag{7}$$

for the top tiling,

$$\text{green} = A_3,\ \text{blue} = G_4,\ \text{red} = E_4,\ \text{ and yellow } = F_4$$

for the second tiling (level-1 supertiles), and

$$\text{green} = C_3,\ \text{blue} = F_2,\ \text{red} = A_2,\ \text{ and yellow } = G_2$$

for the third tiling (level-2 supertiles), one obtains something **which sounds like this**. Note that since the substitution rule from which this was made gives a tiling where all tiles have equal length, there is a common *beat*, emphasizing the rationality of the underlying group. One could give this a time signature of $\frac{5}{5}$.

The left endpoints for the tiling which gives this example is $s \cdot \mathbb{Z}$, where $s > 0$ is defined by the tempo. Recalling Figure 5, we have that $\lambda = 5$ for this example, the bars/measures are level-1 supertiles, or $5s \cdot \mathbb{Z}$. By choosing any $q \in \mathbb{N}$, we can defined the semigroup of beats to be $sq^{-1} \cdot \mathbb{N} \subset s \cdot \mathbb{Q}^+$ making the beat structure of this metre rational.

Before moving to *irrational* metres, let us adopt more terminology and tricks. First, let me point out that in the examples above, we made the simple rule of assigning a specific sound to a particular tile. This applied not only to level-0 tiles (a.k.a. tiles) but also to level-1 and level-2 supertiles. This is the simplest variant of a general composition rule.

Definition 3.1: Let $\mathcal{T}$ be a one-dimensional tiling. A $\mathcal{T}$-equivariant composition rule (or pattern-equivariant composition rule) is a way of assigning sounds to a level-i supertile depending on the level-$(i+j)$ supertile in which it sits, for all $j \leq k$ and some fixed k.

Note that the assignments starting with (7) are $\mathcal{T}$-equivariant for the first example in Figure 5. The name is inspired by $\mathcal{T}$-equivariant functions, which are important functions used in the theory of aperiodic tilings (Kellendonk, 2008). One can consider a $\mathcal{T}$-equivariant composition rule as a $\mathcal{T}$-equivariant function taking values in a set of sounds. The use of sequences in the assignment of sounds, such as those given in Carey and Clampitt (1996), Callender (2013) and Ong (2020), are $\mathcal{T}$-equivariant. The next section gives some tools for giving $\mathcal{T}$-equivariant composition rules in some particular metre, rational or irrational.

4. Writing in $q^{-\ell}s \cdot \mathbb{Z}_n[\lambda]$

So you have a favourite primitive substitution which gives you a primitive substitution matrix M and therefore a metre structured by a group $\Gamma \subset q^{-\ell}s \cdot \mathbb{Z}_n[\lambda]$, for some $\ell, q, n \in \mathbb{N}$ and where λ is the Perron–Frobenius eigenvalue of M. You also know that M does not correspond to a single substitution but to all substitutions whose substitution matrix is M. How can we continue to “compose” in Γ?

There are two immediate tricks that we can use to expand our list of substitutions which will yield metre structures defined by the same group Γ. The first is done by packing more elements into supertiles, while the other allows us to extend the number of symbols while staying in Γ. A third trick, which aims to achieve a phasing effect, can also be done by exploiting the algebraic structure of the group associated to a tiling. It is easy to find variations of each of these tricks, so these are just starting points.

4.1. Trick 1: dilating the inflation

The first one follows the observation that by multiplying the eigenvalue/eigenvector equation $Mv = \lambda v$ by an integer $k > 1$ we obtain the eigenvalue/eigenvector equation $(kM)v = (k\lambda)v$, so that the inflation factor for any substitution rule with substitution matrix kM is $k\lambda$. This new substitution matrix has the same Perron–Frobenius eigenvector v, and so the tile lengths remain the same for the new list of substitution rules with substitution matrix kM. One can also take powers of the substitution matrix Q to obtain matrices for other substitutions with the same group. An example of this is using the matrix M defined by

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{8}$$

which we know satisfies

$$Q^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = M,$$

and so both of these matrices have Perron–Frobenius eigenvector $(\tau, 1)$ and so the associated group here is $\mathbb{Z}[\tau] = \mathbb{Z} + \tau\mathbb{Z}$ (recall that $\tau = (1 + \sqrt{5})/2$). Some readers may recognize the matrix Q as the matrix corresponding to the Fibonacci substitution, which explains the appearance of τ.

4.2. Trick 2: expand and swap

Suppose we are working with substitutions given by (8) but we want to expand the number of symbols while working within the same time signature group. One way to do this would be to expand a substitution given by (8) by, first, taking two copies of M and creating $M^* = M \oplus M$. This, for example, would correspond to a substitution rule formed from two 'disjoint' substitution rules:

$$A \mapsto ABA, \ B \mapsto BA, \ C \mapsto CDC, \ D \mapsto DC. \tag{9}$$

The Perron–Frobenius eigenvector for this substitution is $(\tau, 1, \tau, 1)$, meaning that prototiles A and C have length τ, whereas B and D have length 1. However, any substitution with matrix M^* has two disjoint components, which makes the substitution not primitive. To make it primitive, we can swap letters whose prototiles have the same length which would make the substitution M^* primitive, e.g.

$$A \mapsto ADA, \ B \mapsto BA, \ C \mapsto CBC, \ D \mapsto DC, \tag{10}$$

which is in fact the substitution rule in (2) and we see how it was constructed: we've expanded from substitutions on 2 symbols with group $\mathbb{Z}[\tau]$ to substitutions on 4 symbols with the same group. We can even combine both tricks and use the matrix $M^2 \oplus M^2$

to obtain a substitution

$$A \mapsto ABACDABA, \; B \mapsto BCACB, \; C \mapsto CDCABCDC, \; D \mapsto DACAD \qquad (11)$$

which also has the same group.

4.3. Trick 3: phasing

The use of phasing was popularized in modern Western music through the work of Steve Reich. In this subsection, I will illustrate how the mathematical structure behind metre, in the self-similar case, motivates a couple of ways to obtain a phasing effect.

Phasing usually is done by starting with two voices in unison playing a repetitive melody. As an example, let us assume that the repetitive melody is given by the symbolic string $w_0 = ABACDABA$ and so we can consider the orbit of this string under the simplest of cyclic permutations:

$$\begin{aligned} &w_1 = BACDABAA, w_2 = ACDABAAB, w_3 = CDABAABA, w_4 = DABAABAC, \\ &w_5 = ABAABACD, w_6 = ABAABACD, \text{ and } w_7 = AABACDAB. \end{aligned} \qquad (12)$$

Suppose a note is assigned to each symbol and that all notes have the same length (e.g. quarter notes). Two voices then start playing *ABACDABA* repeatedly and in unison.

The phasing can be done from here in two qualitatively different ways:

Phasing in space (a.k.a. phase shifting) This type of phasing is achieved by having the second voice going through the cyclical permutations of the starting phrase while the first voice repeats the starting phrase. For example, for the symbolic strings w_i above, if we denote the top line by the first voice and the bottom line by the second voice, this would look something like

$$\begin{matrix} w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0w_0 \\ w_0w_0w_0w_0w_1w_1w_2w_2w_3w_3w_3w_4w_4w_4w_5w_5w_6w_6w_7w_7w_7w_7w_0w_0w_0w_0, \end{matrix}$$

which is the type of structure that the first part of Steve Reich's *Piano Phase* has. If w_0' is another symbolic string of the same length as w_0, then a similar thing can be done by going through the cyclic permutations of w_0' over w_0, which is how the second part of Reich's *Piano Phase* is put together.

Phasing in time (a.k.a. rhythm shifting) This is achieved by having the second voice change its tempo so that the two voices become out of phase.

Let me now explain how we can approach both types of phasing using the constructions from above.

4.3.1. Phase shifting

The easiest thing one can do to obtain phasing in space is to shift a repetitive tiling $\mathcal{T}$, along with the metre it defines, by an element $\Gamma^+_{\mathcal{T}}$. This will obtain a phase shifting effect.

A more interesting phase shifting can be obtained as follows: starting with one primitive symbolic substitution rule one can make a minor local modification of it to obtain a second one. By minor modification, I mean change the order of at least two letters which appear in the substitution of a symbol. For example, consider the following two symbolic substitution rules, the first of which was created using the same technique used to construct (11):

$$\begin{array}{rcl} A & \mapsto & ABACBAFA \\ B & \mapsto & ADCBA \\ C & \mapsto & AFDBCECA \\ D & \mapsto & CDEFA \\ E & \mapsto & ACEFDBCA \\ F & \mapsto & DCBCA \end{array} \quad \text{and} \quad \begin{array}{rcl} A & \mapsto & ABACBAFA \\ B & \mapsto & ADCBA \\ C & \mapsto & BCECAAFD \\ D & \mapsto & CDEFA \\ E & \mapsto & ACEFDBCA \\ F & \mapsto & CBCAD \end{array}$$

Note that the second is obtained from the first by permuting the order of some of the strings obtained from the substitution. This local change, as the substitution rule gets iterated, makes the symbolic strings become "out of phase" every now and then. Making a pattern-equivariant choice of sounds for the first substitution and superimposing the two (meant to come out of phase occasionally), **sounds like this**.

4.3.2. Rhythm shifting

Taking the metre defined by some self-similar tiling $\mathcal{T}$ with expansion constant $\lambda > 1$, there is a natural way to do rhythm shifting. Since this is done by modifying the tempo of one of the voices, and since the tempo is defined by the frequency of a tiling, rhythm shifting can be obtained by changing the frequency of a tiling, and this is done by deforming the tiling and changing the geometry of some of its tiles.

Here is a concrete way to do so using the expansion factor λ. Let $\Lambda_{\mathcal{T}} = \{p_i\}_i \subset \mathbb{R}$ be the set defined by the left endpoints of tiles in $\mathcal{T}$, assuming without loss of generality that $p_0 = 0$. Pick $0 < p_{n_1} < p_{n_2} < p_{n_3}$ in $\Lambda_{\mathcal{T}}$. Let $f : [p_{n_1}, p_{n_2}] \to [p_{n_1}, p_{n_3}]$ be a function such that

$$\begin{aligned} &f(p_{n_1}) = p_{n_1}, \quad f'(p_{n_1}) = 1, f'(x) \in (1, \lambda) \text{ for } x \in (p_{n_1}, p_{n_2}), \\ &f(p_{n_2}) = p_{n_3}, \quad \text{and} \quad f'(p_{n_2}) = \lambda. \end{aligned} \tag{13}$$

Then such a function induces a deformation the tiling $\mathcal{T}$ to obtain a new tiling $\mathcal{T}' = \Phi(\mathcal{T})$ by sending the set of left endpoints $\Lambda_{\mathcal{T}}$ to the set of left endpoints $\Lambda_{\mathcal{T}'} = \Phi(\Lambda_{\mathcal{T}})$ of $\mathcal{T}'$ defined by

$$\Phi(x) = \begin{cases} x & \text{if } x \leq p_{n_1} \\ f(x) & \text{if } x \in [p_{n_1}, p_{n_2}] \\ \lambda x + p_{n_3} - \lambda p_{n_2} & \text{if } x \geq p_{n_2} \end{cases}$$

A function $f : [p_{n_1}, p_{n_2}] \to [p_{n_1}, p_{n_3}]$ which achieves this is the function

$$f(x) = \int_0^x 1 + \frac{(s - p_{n_1})^\kappa}{(p_{n_2} - p_{n_1})^\kappa}(\lambda - 1)\, \mathrm{d}s \quad \text{with } \kappa = \frac{(p_{n_2} - p_{n_1})(\lambda - 1)}{p_{n_3} - p_{n_2}} - 1,$$

which the reader may verify satisfies (13). The relationship between $\mathcal{T}$ and its deformation $\mathcal{T}'$ are

(i) The tilings $\mathcal{T}$ and $\mathcal{T}'$ are indistinguishable before the point $x = p_{n_1}$,
(ii) The tiles between p_{n_1} and p_{n_3} in $\mathcal{T}'$ are a continuous deformation of the tiles between p_{n_1} and p_{n_2} in $\mathcal{T}$, scaling the geometry of level-0 supertiles to that of level-1 supertiles,

(iii) The tiles of the deformed tiling $\mathcal{T}'$, after $x = p_{n_3}$, have the geometry of level-1 supertiles of the tiling $\mathcal{T}$, as they are scaled by λ.

As such, the tempo of anything played based on $\mathcal{T}'$ *slows down* after p_{n_1}. One can do a similar construction which speeds up tempo by deforming a tiling which, after certain point, brings the geometry of level-k supertiles to that of level-$(k-1)$ supertiles. The composition of two deformations, one which speeds up the tempo and one which slows it down, has the overall effect of phase shifting, i.e.shifts in space.

4.4. Some examples

This track takes the substitution in (11) and lays down sounds on a pattern-equivariant way. Another take on the same substitution is given by this track, which takes the above substitution and lays drums and chords in a pattern-equivariant way. Note that since the relevant group for the substitution (11) contains a scalar multiple of $\mathbb{Z}[\tau]$, giving it a truly irrational metre.

Another example of the construction above is the following. Consider the real solution (and largest in modulus) of the polynomial equation $x^3 - x - 1 = 0$. This is the so-called plastic ratio

$$\mathfrak{p} = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}.$$

The companion matrix[8] for that polynomial is

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

which has a Perron–Frobenius eigenvector of the form $(\mathfrak{p}_*, \mathfrak{p}, 1)$ with eigenvalue $\mathfrak{p}$, for some algebraic number $\mathfrak{p}_*$. We now employ the same trick which we used to obtain (10) from (9) to double the number of symbols while keeping the tile lengths in the same group defined by the plastic ratio. That is, there is a way of applying Trick 2 from Section 4.2 in such a manner to obtain the following matrix

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, the matrix $\mathcal{S}^6$ is the substitution matrix for the following symbolic substitution

$$\begin{array}{lll} A \mapsto AFBF, & B \mapsto BCDCECD, & C \mapsto CAEFE \\ D \mapsto DCEC, & E \mapsto EFAFABF, & F \mapsto FBCBD, \end{array}$$

which has expansion factor $\mathfrak{p}^6$. Assigning sounds in a pattern-equivariant way, we obtain this track.

Note that in this last example, I started with a specific algebraic number 𝔭 and constructed a substitution rule related to it. This generalizes easily and provides a way to write in an algebraic metre related to the real root of any polynomial $P(x) = x^n - a_n x^{n-1} - \cdots - a_1 x - a_0$ with integers $a_i \geq 0$ as long as the companion matrix for $P(x)$ is primitive. This generalizes further for any Perron number, see Lind (1987, § 3).

Notes

1. Since I am completely ignorant of the structure of other musical traditions, I am narrowing my claim here in acknowledgement of this ignorance and not claiming that these properties are unique to the Western musical tradition. I would be delighted to learn whether or not the ideas here can also be adapted to non-Western musical systems.
2. It has also been claimed, and strongly disputed, that the intricate artwork in Islamic architecture over centuries and over a large geographic area has underlying aperiodic structure (Bonner, 2017; Cromwell, 2015).
3. Sturmian sequences are non-periodic sequences with the lowest possible complexity.
4. This is not *the* chair tiling, that is, it is not the substitution rule which is called "the chair tiling" in the tiling literature, but a variation. The canonical chair tiling has an expansion factor of 2 whereas the variation here has an expansion factor of 4.
5. The curious reader may want to verify the following: after n steps applying the inflation and substitution rule, we obtain patches which are tiled by roughly λ^{2n} tiles.
6. Note that if a letter is substituted into a word which begins with the same letter, the words obtained by applying the substitution rule stabilize.
7. Some people follow the convention that would make this matrix the transpose of the substitution matrix.
8. A companion matrix M_p for a polynomial $p(x)$ is the matrix A such that its characteristic polynomial is $p(x)$. One can check that $x^3 - x - 1$ is the characteristic polynomial os S above.

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ORCID

Rodrigo Treviño http://orcid.org/0000-0001-7818-5103

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