TRACES OF RANDOM OPERATORS ASSOCIATED WITH SELF-AFFINE DELONE SETS AND SHUBIN'S FORMULA

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Abstract. We study operators defined on a Hilbert space defined by a self-affine Delone set Λ and show that the usual trace of a restriction of the operator to finite-dimensional subspaces satisfies a certain $\limsup$ law controlled by traces on a certain subalgebra. The asymptotic traces are defined through asymptotic cycles, or $\mathbb{R}^d$-invariant distributions of a dynamical system defined by Λ. We use this to refine Shubin's trace formula for self-adjoint operators and show that the errors of convergence in Shubin's formula are given by these traces.

1. Introduction and statement of results

This paper is about traces on algebras of random operators associated to dynamical systems defined by aperiodic, self-affine Delone sets. Algebras of operators close to the ones studied here have been considered before different authors (e.g. [Kel95, BHZ00, LS03]) and they are motivated in part by the study of spectral properties of Schrödinger operators arising in the study of quasicrystals. Since quasicrystals are modeled by Delone sets Λ, which are uniformly discrete subsets of $\mathbb{R}^d$, this leads to the study of self-adjoint operators on Hilbert spaces of the form $\ell^2(\Lambda)$ defined by Λ.

More specifically, let A be a self-adjoint operator on $\ell^2(\Lambda)$ and $A|_{B_T}$ be the restriction of A to the subspace $\ell^2(\Lambda \cap B_T)$, where B_T is a ball of radius T around the origin. For any $E > 0$, define

$$n_T^A(E) = \frac{\#\{\text{ eigenvalues of } A \text{ less than or equal to } E\}}{\mathrm{Vol}(B_T)}.$$

The function $E \mapsto n_T^A(E)$ is a distribution function of the measure ρ_T^A defined as

$$\int \varphi \rho_T^A = \frac{\mathrm{tr}\left(\varphi(A|_{B_T})\right)}{\mathrm{Vol}(B_T)}$$

for $\varphi \in C_0(\mathbb{R})$. It turns out that there exists a unique trace $\tau : C^*(\Lambda) \to \mathbb{C}$ such that $\rho_T^A(\varphi) \to \rho_A(\varphi)$ in the weak topology, where the measure ρ_A is defined as $\rho_A(\varphi) = \tau(\varphi(A))$ and $C^*(\Lambda)$ is a C*-algebra defined by Λ (see e.g. [LS03] and references therein). The limiting distribution of the measure ρ_A is called the **integrated density of states**. Shubin's trace formula

$$\lim_{T\to\infty} \frac{1}{\mathrm{Vol}(B_T)} \mathrm{tr}\left(\varphi(A|_{B_T})\right) = \tau(\varphi(A)), \tag{1}$$

which asserts that the trace per unit volume gives the integrated density of states, gives information about the structure of the spectrum $\sigma(A)$. The goal of this paper is to refine the convergence in (1). For example, we want to know whether there is an asymptotic statement that can be made for the difference

$$\mathrm{tr}\left(\varphi(A|_{B_T})\right) - \rho_A(\varphi)\mathrm{Vol}\left(B_T\right)$$

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as $T \to \infty$. In this case, any statement would yield information on the error rates of the integrated density of states given by self-adjoint operator A. By applying the results of [ST15], in this paper, we will show that as long as the quasicrystal is self-affine, the convergence in Shubin's formula (1) can be refined.

The route to this refinement is the study of deviations of ergodic averages of uniquely ergodic $\mathbb{R}^d$-actions on a compact metric space Ω_Λ defined by a self-affine Delone set Λ. The types of Delone sets which are studied are called **renormalizable of finite type** (RFT Delone sets), which are defined in §3.1. Examples of such Delone sets are given by aperiodic substitution tilings such as the Penrose tilings and the Ammann-Beenker tilings in two dimensions, and the icosahedral tilings in three dimensions. There are also examples coming from the cut and project construction which are not immediately given by substitution tilings.

We briefly summarize the setup from [ST15] relevant in order to state the main results of this paper (see §5 for details). Given any RFT Delone set Λ, there exist $d_\Lambda \geq 1$ numbers $\nu_1 > |\nu_2| \geq \cdots \geq |\nu_{d_\Lambda}| > 1$ which are the eigenvalues of an induced map on the cohomology space $H^d(\Omega_\Lambda;\mathbb{R})$ (this is defined in §3). Along with these numbers are cohomology classes $[\eta_{i,j,k}]$ which are generalized eigenvectors of the induced action on cohomology and are represented by d-forms $\eta_{i,j,k}$ on $\mathbb{R}^d$. These classes are indexed by a set I^+_Λ. There are dual currents $\mathfrak{C}_{i,j,k}$ such that, roughly speaking, if $\mathfrak{C}_{i,j,k}(\eta) \neq 0$ then $\int_{B_T}\eta$ is at least of the order of T^{ds_i}, where $s_i = \frac{\log|\nu_i|}{\log|\nu_1|}$. The set I^+_Λ is partially ordered: $(i,j,k) \leq (i',j',k')$ if $L(i,j,T)T^{ds_i} \geq L(i',j',T)T^{ds_{i'}}$ for any (and all) $T > 1$, where $L(i,j,T)$ are some given non-negative integer powers of $\log T$ (the order does not depend on the indices k).

While the leading current $\mathfrak{C}_{1,1,1}$ represents the unique $\mathbb{R}^d$-invariant measure for the translation action on Ω_Λ, the other currents capture a dynamical invariant for smooth functions. From the invariant currents $\mathfrak{C}_{i,j,k}$ we obtain traces $\tau_{i,j,k}$ on a certain $*$-algebra $\mathcal{A}^{tlc}_\Lambda \subset C^*(\Lambda)$ of bounded operators on $\ell^2(\Lambda)$ called **Λ-equivariant operators of finite range** (see Proposition 1). This $*$-algebra of operators contain many operators of interest to physics, such as Hamiltonian operators of the form $H = -\triangle + V$, for Λ-equivariant potentials V. As such, the trace $\tau_{1,1,1}$ is the trace τ from Shubin's trace formula (1). In what follows, given a bounded set $B_0 \subset \mathbb{R}^d$, B_T is a one-parameter family of sets obtained from B_0 by multiplying by a one parameter family of matrices g_T in such a way that $\mathrm{Vol}(B_T) = \mathrm{Vol}(B_0)T^d$. We now give our refinement to Shubin's formula.

Theorem 1. *Let Λ be an RFT Delone set, B_0 a Lipschitz domain and $A \in \mathcal{A}^{fin}_\Lambda$ a self-adjoint operator. For any index (i,j,k) there exists a regular countably additive Borel measure $\rho^A_{i,j,k}$ such that for any polynomial $\varphi \in C(\mathbb{R})$,*

(2)
$$\limsup_{T\to\infty}\frac{1}{L(i,j,T)T^{d\frac{\log|\nu_i|}{\log\nu_1}}}\left(\mathrm{tr}(\varphi(A|_{B_T})) - \sum_{\substack{(i',j',k')\leq(i,j,k)\\ k'\neq k}}\rho^A_{i',j',k'}(\varphi)\Psi^{B_0}_{i',j',k'}(T)L(i',j',T)T^{d\frac{\log|\nu_{i'}|}{\log\nu_1}}\right) = \rho^A_{i,j,k}(\varphi),$$

where $L(i,j,T)$ is a non-negative integer power of $\log T$ and $\Psi^{B_0}_{i,j,k} : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function satisfying $\limsup_{T\to\infty}\Psi^{B_0}_{i,j,k}(T) = 1$. The measures are defined by $\rho^A_{i,j,k}(\varphi) = \tau_{i,j,k}(\varphi(A))$, where $\tau_{i,j,k}$ are traces on $\mathcal{A}^{tlc}_\Lambda$.

Theorem 1 gives rates of convergence to Shubin's formula (1) in the case φ is a polynomial, that is, errors of the integrated density of states: the distributions of the measures $\rho^A_{i,j,k}$ capture the error in the convergence to the integrated density of states. We do not know what physical interpretation the measures $\rho^A_{i,j,k}$ or their distributions may have. We do not know, at the moment, how to extend the $\limsup$ statement (2) in Theorem 1 to all continuous functions $\varphi \in C(\mathbb{R})$ (see Remark 7), although the functionals $\rho^A_{i,j,k}$ are measures. A more general theorem on traces on the $*$-algebra of operators $\mathcal{A}^{tlc}_\Lambda$ can also be derived from the results of [ST15].

Theorem 2. *Let Λ be an RFT Delone set and B_0 a Lipschitz domain. For every index $(i,j,k) \in I^+_\Lambda$ there exists a trace $\tau_{i,j,k} : \mathcal{A}^{tlc}_\Lambda \to \mathbb{C}$ such that for any $A \in \mathcal{A}^{fin}_\Lambda$*

(3)

$$\limsup_{T\to\infty} \frac{1}{L(i,j,T)T^{d\frac{\log|\nu_i|}{\log|\nu_1|}}} \left(\mathrm{tr}(A|_{B_T}) - \sum_{\substack{(i',j',k')\leq(i,j,k)\\ k'\neq k}} \tau_{i',j',k'}(A)\Psi^{B_0}_{i',j',k'}(T)L(i',j',T)T^{d\frac{\log|\nu_{i'}|}{\log \nu_1}} \right) = \tau_{i,j,k}(A),$$

where $L(i,j,T)$ is a non-negative power of $\log T$ and $\Psi^{B_0}_{i,j,k} : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function satisfying $\limsup_{T\to\infty} \Psi^{B_0}_{i,j,k}(T) = 1$.

It is important to point out that the traces $\tau_{i,j,k}$, defined on a dense $*$-subalgebra $\mathcal{A}^{tlc}_\Lambda$ of the C*-algebra $C^*(\Lambda)$, **do not extend to the full C*-algebra**, with the exception of $\tau_{1,1,1}$, which comes from the unique $\mathbb{R}^d$-invariant measure for the translation action on Ω_Λ. This is due to the fact that the currents $\mathfrak{C}_{i,j,k}$ from which the traces are defined are functionals defined only on forms possessing sufficient regularity, not just continuous forms. As such, we need extra regularity for functions which are represented by operators, which leads to operators which define functions on Ω_Λ which have the property of being transversally locally constant[1]. See Remark 6.

As far as we are aware, the results above are the first ones which relate traces of self-adjoint operators to more than one homology class (closed cycles) of $\mathrm{Hom}(H^d(\Omega_\Lambda;\mathbb{R});\mathbb{R})$. In recent years there has been a string of results for one-dimensional self-similar tilings, amongst which the Fibonacci Hamiltonian, defined by the Fibonacci substitution, has been thoroughly studied (see [DEG15] for a comprehensive survey of the results). Many of the results on the spectral properties of Schrödinger operators concern ones coming from Sturmian substitutions. It is known that in such case, $\dim H^1(\Omega_\Lambda;\mathbb{R}) = 2$ and that $|I^+_\Lambda| = 1$, so our theorems do not say anything about Shubin's formula in such contexts; there is no refinement in those cases. Since the cohomology of tilings and Delone sets capture some sort of complexity, the Sturmian substitutions are not complex enough to have non-trivial expansions in (2) and (3). However, it is easy to construct substitutions on more than 2 symbols for which there are expansions for our main theorems. For higher dimensions, operators coming from well-known substitution tilings such as the Penrose or Ammann-Beenker tilings do admit a refinement since $|I^+_\Lambda| = 3$ in those cases.

This paper is organized as follows. In section 2 we recall Delone sets, pattern spaces and the translation action on pattern spaces. In section 3 we go over the necessary definitions of cohomology for pattern spaces as well as RFT Delone sets. In section 4 we recall the algebras of

[1] We thank I. Putnam for pointing this out to us.

operators with which we will work and relate them to the cohomology spaces defined in the previous section. In section 5 we recall the relevant results of [ST15], show that we can define traces from $\mathbb{R}^d$-invariant distributions on the pattern space from [ST15], and prove the main theorems.

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2. Delone sets

A subset $\Lambda \subset \mathbb{R}^d$ is a **Delone set** if it satisfies two conditions:

(i) **Uniformly discrete:** There exists a $r > 0$ such that any distinct two points $x, y \in \Lambda$ are separated by a distance of at least r;

(ii) **Relatively dense:** There exists an $R > 0$ such that any ball of radius $R > 0$ such that for any point $x \in \mathbb{R}^d$, the ball of radius R centered at x contains at least one other point of Λ.

The radius r involved in uniform discreteness is called the **packing radius**; the radius R involved in relative density is called the **covering radius**. A **cluster** is a finite subset of Λ. A Delone set has **finite local complexity** if for any given $R > 0$, the set of all clusters found in any ball of radius R, up to translation, is finite.

For any Delone set Λ, we denote by $\varphi_t(\Lambda)$ the translation of the set Λ by the vector $t \in \mathbb{R}^d$. For any two translates Λ, Λ' of Λ, we define the distance between them by

$$d(\Lambda, \Lambda') = \inf\{\varepsilon > 0 : B_{\varepsilon^{-1}}(0) \cap \varphi_x(\Lambda) = B_{\varepsilon^{-1}}(y) \cap \varphi_y(\Lambda') \text{ for some } x, y \in B_\varepsilon(0)\}.$$

The **pattern space of** Λ is the closure of the set of translates of Λ with respect to the above metric:

$$\Omega_\Lambda = \overline{\{\varphi_t(\Lambda : t \in \mathbb{R}^d)\}}.$$

The **canonical transversal** of the pattern space Ω_Λ is the set

$$\mho_\Lambda = \{\Lambda' \in \Omega_\Lambda : \bar{0} \in \Lambda'\}.$$

If Λ has finite local complexity, the canonical transversal $\mho_\Lambda$ is a Cantor set. In that case, the topology induced by the metric on $\mho_\Lambda$ is generated by clopen sets given by specifying clusters of Λ. That is, any given cluster $C \subset \Lambda$ and a point $p \in C$, the set $U_C \subset \mho_\Lambda$ given by all patterns $\Lambda' \in \Omega_\Lambda$ with a cluster equivalent to C around the origin (and p identified to the origin) is a clopen set in the topology.

Thus, when Λ has finite local complexity, the pattern space Ω_Λ has a local structure modeled on sets of the form $B_\varepsilon \times \mathcal{C}$ where $B_\varepsilon \subset \mathbb{R}^d$ is an open ball and $\mathcal{C}$ is a Cantor set. The pattern space is a foliated space where the leaves of the foliation are orbits of the $\mathbb{R}^d$ action.

The only types of Delone sets which will be considered here are those which give rise to a uniquely ergodic $\mathbb{R}^d$-action on Ω_Λ, that is, sets for which there is a unique $\mathbb{R}^d$-invariant probability measure μ on Ω_Λ which is invariant under the translation action φ_t of $\mathbb{R}^d$. Moreover, for any $f \in C(\Omega_\Lambda)$ and any Følner sequence $\{B_T\}$,

$$\frac{1}{\mathrm{Vol}(B_T)} \int_{B_T} f \circ \varphi_t(\Lambda_0)\, dt \longrightarrow \int_{\Omega_\Lambda} f(\Lambda')\, d\mu(\Lambda')$$

uniformly for any Λ_0. By the local product structure of Ω_Λ, the invariant measure has a local product structure of the form $\mu = \mathrm{Leb} \times \mathfrak{m}_\Lambda$, where $\mathfrak{m}_\Lambda$ is a probability measure on $\mho_\Lambda$. Given a cluster $C \subset \Lambda$, the measure $\mathfrak{m}_\Lambda(U_C)$ is the asymptotic frequency of the cluster C in Λ [LMS02]. These systems are always minimal in the sense that every leaf of the foliation of Ω_Λ is dense in Ω_Λ.

3. Cohomology

Let $\Lambda \subset \mathbb{R}^d$ be a Delone set.

Definition 1. A continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is Λ**-equivariant** [Kel03] if there exists an $R > 0$ such that

$$B_R(x) \cap \Lambda = B_R(y) \cap \Lambda \text{ implies } f(x) = f(y).$$

Differential forms which are Λ-equivariant are defined as differential forms for which the functions are Λ. We denote by Δ^k_Λ the set of all C^∞ k-forms which are Λ-equivariant. The complex $0 \to \Delta^0_\Lambda \to \Delta^1_\Lambda \to \cdots \to \Delta^d_\Lambda \to 0$ is a subcomplex of the de Rham complex.

Definition 2. The k^{th} Λ-equivariant cohomology spaces are defined by

$$H^k(\Omega_\Lambda; \mathbb{R}) = \frac{\ker \{d : \Delta^k_\Lambda \to \Delta^{k+1}_\Lambda\}}{\mathrm{Im}\, \{d : \Delta^{k-1}_\Lambda \to \Delta^k_\Lambda\}}.$$

The set $C^\infty_{tlc}(\Omega_\Lambda)$ is the set of **transversally locally constant functions**, that is, the set of continuous functions on Ω_Λ which are (C^∞) smooth in the leaf direction of the foliation and locally constant in the transverse direction. For any such function, for any Λ there exists an $R > 0$ such that if for any Λ' with $\Lambda \cap B_R(0) = \Lambda' \cap B_r(0)$, then $f(\Lambda) = f(\Lambda')$. By [KP06] there is an algebra isomorphism $i_\Lambda : C^\infty_{tlc}(\Omega_\Lambda) \to \Delta^0_\Lambda$ given, for any $h \in C^\infty_{tlc}(\Omega_\Lambda)$, by

$$f_h(t) := i_\Lambda(h)(t) = h \circ \varphi_t(\Lambda). \tag{4}$$

By the local product structure, any $h \in C^\infty_{tlc}(\Omega_\Lambda)$ defines a locally constant function on the canonical transversal $g_h : \mho_\Lambda \to \mathbb{R}$. We denote the set of locally constant functions on $\mho_\Lambda$ by $C^\infty(\mho_\Lambda)$.

To any Λ-equivariant function (and by the isomorphism above, to any function in $C^\infty_{tlc}(\Omega_\Lambda)$) we can assign a unique cohomology class in $H^d(\Omega_\Lambda; \mathbb{R}^d)$. This is done as follows: let $f \in \Delta^0_\Lambda$ and denote by $(\star 1)$ the smooth volume form in $\mathbb{R}^d$. Then $f(\star 1)$ is a closed form in Δ^d_Λ and therefore it has a cohomology class $[f(\star 1)]$. The **cohomology class of** f is defined to be $[f(\star 1)] \in H^d(\Omega_\Lambda; \mathbb{R}^d)$.

Given a locally constant function $g \in C^\infty(\mho_\Lambda)$ we can extend it to a transversally locally constant function $h \in C^\infty_{tlc}(\Omega_\Lambda)$ by "smoothing" the function along the foliation direction in Ω_Λ. By the isomorphism (4), this extension defines a Λ-equivariant function $f_g \in \Delta^0_\Lambda$. As such, we can define the cohomology class of $g \in C^\infty(\mho_\Lambda)$ to be the cohomology class $[f_g]$ of f_g. Thus, characteristic functions $\chi_U \in C^\infty(\mho_\Lambda)$ of clopen sets $U \subset \mho_\Lambda$ have cohomology classes and linear functionals on $C^\infty(\mho_\Lambda)$ can be identified with linear functionals on $H^d(\Omega_\Lambda; \mathbb{R})$, that is, with homology classes.

3.1. Self-affinity.

We recall some definitions from [ST15]. Let $\Lambda \subset \mathbb{R}^d$ be a Delone set for which the $\mathbb{R}^d$ action on Ω_Λ is uniquely ergodic and denote by μ the unique $\mathbb{R}^d$-invariant measure. In that case, Λ is **renormalizable of finite type** (RFT) if

(i) There exists an expanding matrix $A \in GL^+(d, \mathbb{R})$ and a μ-preserving homeomorphism $\Phi_A : \Omega_\Lambda \to \Omega_\Lambda$ satisfying the conjugacy

$$\Phi_A \circ \varphi_t = \varphi_{At} \circ \Phi_A \tag{5}$$

for any $t \in \mathbb{R}^d$. By expanding, we mean that A has all eigenvalues, in modulus, greater than one.

(ii) The spaces $H^*(\Omega_\Lambda;\mathbb{R})$ are finite dimensional.

Remark 1. There are plenty RFT Delone sets. For example, substitution tilings, by [AP98], have finite dimensional cohomology spaces. Moreover, the substitution action induces the μ-preserving homeomorphism of Ω_Λ. Cut and project Delone sets can also be constructed to be RFT [ST15, §8].

For an RFT Delone set Λ there is an induced action $\Phi_A^* : H^d(\Omega_\Lambda;\mathbb{R}) \to H^d(\Omega_\Lambda;\mathbb{R})$. Since $H^*(\Omega_\Lambda;\mathbb{R})$ is finite dimensional, we list the finite set of eigenvalues in decreasing order by norm: $\nu_1 > |\nu_2| \geq \cdots \geq |\nu_r|$. We also list the eigenvalues of A by norm: $|\lambda_1| \geq \cdots \geq |\lambda_d| > 1$.

Let E_i be the generalized eigenspaces for the action of Φ_A^* on $H^d(\Omega_\Lambda;\mathbb{R})$ induced by the map by Φ_A corresponding to the eigenvalue ν_i. The subspaces E_i are decomposed as

$$E_i = \bigoplus_{j=1}^{\kappa(i)} E_{i,j},$$

where $\kappa(i)$ is the size of the largest Jordan block associated with ν_i and the $E_{i,j}$ is the span of vectors corresponding to columns in Jordan blocks J with with property that $J_{j,j} = \nu_i$ and $J_{j,j-1} = 1$. For each i, we choose a basis of classes $\{[\eta_{i,j,k}]\}$ with the property that $\langle [\eta_{i,j,1}], [\eta_{i,j,2}], \ldots, [\eta_{i,j,s(i,j)}]\rangle = E_{i,j}$ and

$$\mathcal{A}_d[\eta_{i,j,k}] = \nu_i[\eta_{i,j,k}] + [\eta_{i,j-1,k}]. \tag{6}$$

Definition 3. The **rapidly expanding subspace** $E_A^+(\Omega_\Lambda) \subset H^d(\Omega_\Lambda;\mathbb{R})$ is the direct sum of all generalized eigenspaces E_i of $\mathcal{A}_d$ such that the corresponding eigenvalues ν_i of Φ_A^* satisfy

$$\frac{\log|\nu_i|}{\log \nu_1} \geq 1 - \frac{\log|\lambda_d|}{\log \nu_1}. \tag{7}$$

We set $I_\Lambda^+ = I_\Lambda^{+,>} \cup I_\Lambda^{+,=}$ be the index set of classes $[\eta_{i,j,k}]$ which form a generalized eigenbasis for E_Λ^+, where the indices in $I_\Lambda^{+,>}$ contains vectors corresponding to a strict inequality in (7) and the indices in $I_\Lambda^{+,=}$ correspond to vectors associated to eigenvalues which give an equality in (7). Note that $I_\Lambda^{+,=}$ can be empty but $I_\Lambda^{+,>}$ always has at least one element.

Let $\eta_{i,j,k} \in \Delta_\Lambda^d$ be a representative of the class $[\eta_{i,j,k}]$ in the eigenbasis in (6). By [ST15, §4], there exist forms $\zeta_{i,j,k} \in \Delta_\Lambda^{d-1}$ such that

$$A^*\eta_{i,j,k} = \nu_i\eta_{i,j,k} + \eta_{i,j-1,k} + d\zeta_{i,j,k}. \tag{8}$$

For any $\eta \in \Delta_\Lambda^d$, we denote by $\alpha_{i,j,k}(\eta)$ the component of the class $[\eta]$ in the subspace spanned by $[\eta_{i,j,k}]$. In other words, $\eta = \sum_{i,j,k} \alpha_{i,j,k}(\eta)\eta_{i,j,k} + d\omega_\eta$ for some $\omega_\eta \in \Delta_\Lambda^{d-1}$. For any $f \in \Delta_\Lambda^0$ the component $\alpha_{i,j,k}(f)$ is defined by duality: $\alpha_{i,j,k}(f) := \alpha_{i,j,k}(f(\star 1))$.

In [ST15] we showed that the dual space $\mathfrak{C}_\Lambda^+ = (E_\Lambda^+)'$ to the rapidly expanding subspace E_Λ^+ is represented by $\mathbb{R}^d$-invariant Λ-equivariant currents. It admits a decomposition into eigenvectors of the induced action by Φ_A:

$$\mathfrak{C}_\Lambda^+ = \bigoplus_{(i,j,k)\in I_\Lambda^+} \operatorname{span} \mathfrak{C}_{i,j,k}, \tag{9}$$

where the $\mathfrak{C}_{i,j,k} \in (\Delta^d_\Lambda)'$. We can identify $\mathfrak{C}^+_\Lambda$ with a subspace of the homology of Ω_Λ. The currents $\mathfrak{C}_{i,j,k} \in \mathfrak{C}^+_\Lambda$ define distributions $\Gamma_{i,j,k}$ on $C^\infty(\mho_\Lambda)$ as follows. Recall that a function $g \in C^\infty(\mho_\Lambda)$ has a cohomology class $[g] \in H^d(\Omega_\Lambda;\mathbb{R})$. The distributions $\Gamma_{i,j,k}$ are defined as

$$\Gamma_{i,j,k}(g) := \mathfrak{C}_{i,j,k}([g])$$

for any $g \in C^\infty(\mho_\Lambda)$. We point out that we have that $\Gamma_{1,1,1} = \mathfrak{m}_\Lambda$, where $\mathfrak{m}_\Lambda$ is the canonical measure on $\mho_\Lambda$ coming from the $\mathbb{R}^d$-invariant measure on Ω_Λ (this will be explained in §5).

4. Algebras and cohomology

In this section we recall the setup from [LS03] which will allow us to work with operators of the right kind. Let

$$\mathcal{G}_\Lambda(g) = \{(p,\Lambda',q) \in \mathbb{R}^d \times \Omega_\Lambda \times \mathbb{R}^d : p,q \in \Lambda'\}.$$

Definition 4. A **kernel of finite range** is a function $k \in C(\mathcal{G}_\Lambda)$ such that:

(i) k is bounded;
(ii) k has finite range: there exists an $R_k > 0$ such that $k(p,\Lambda',q) = 0$ whenever $|p-q| \geq R_k$;
(iii) k is $\mathbb{R}^d$-invariant: for any $t \in \mathbb{R}^d$ we have that $k(p+t,\Lambda'+t,q+t) = k(p,\Lambda',q)$.

The set of all kernels of finite range is denoted by $\mathcal{K}^{fin}_\Lambda$. Note that for any two $\Lambda_1,\Lambda_2 \in \Omega_\Lambda$ we have that $\mathcal{K}^{fin}_{\Lambda_1} = \mathcal{K}^{fin}_{\Lambda_2}$. For any kernel $k \in \mathcal{K}^{fin}_\Lambda$ there is a family of representations $\{\pi_{\Lambda'}\}_{\Lambda' \in \Omega_\Lambda}$ in $\mathcal{B}(\ell^2(\Lambda'))$ defined, for $k \in \mathcal{K}^{fin}_\Lambda$ by

$$\langle K_{\Lambda'}\delta_p,\delta_q\rangle := \langle (\pi_{\Lambda'}k)\delta_p,\delta_q\rangle = k(p,\Lambda',q)$$

for $p,q \in \Lambda'$.

The family $\{K_{\Lambda'}\}_{\Lambda' \in \Omega_\Lambda}$ is bounded in the product $\Pi_{\Lambda' \in \Omega_\Lambda}\mathcal{B}(\ell^2(\Lambda'))$. To make $\mathcal{K}^{fin}_\Lambda$ a $*$-algebra, the convolution product is defined as

$$(a\cdot b)(p,\Lambda',q) = \sum_{x \in \Lambda'} a(p,\Lambda',x)b(x,\Lambda',q)$$

and the involution by $k^*(p,\Lambda',q) = \bar{k}(q,\Lambda',p)$. As such, the map $\pi : \mathcal{K}^{fin}_\Lambda \to \Pi_{\Lambda' \in \Omega_\Lambda}\mathcal{B}(\ell^2(\Lambda'))$ is a faithful $*$-representation. The image of this map is denoted by $\mathcal{A}^{fin}_\Lambda$ and it is the algebra of **operators of finite range**. The completion of this space under the norm $\|A\| = \sup_{\Lambda' \in \Omega_\Lambda}\|A_{\Lambda'}\|$ is denoted by $\mathcal{A}_\Lambda$.

Remark 2. As shown in [LS03], the algebra $\mathcal{A}_\Lambda$ is closely related (in fact, $*$-isomorphic) to the algebras considered in [BHZ00, Kel95].

Definition 5. The set of Λ-**equivariant kernels of finite range** are the kernels of finite range $k \in \mathcal{K}^{fin}_\Lambda$ for which there exists a $R_k > 0$ such that if for two $\Lambda_1,\Lambda_2 \in \Omega_\Lambda$ we have that $B_{R_k}(0)\cap\Lambda_1 = B_{R_k}(0)\cap\Lambda_2$ then for any $p,q \in B_{R_k}(0)\cap\Lambda_1$ we have that $k(p,\Lambda_1,q) = k(p,\Lambda_2,q)$.

We denote the set of Λ-equivariant kernels by $\mathcal{K}^{tlc}_\Lambda$ which is a $*$-subalgebra of $\mathcal{K}^{fin}_\Lambda$. The image of Λ-equivariant kernels $\mathcal{A}^{tlc}_\Lambda = \pi\mathcal{K}^{tlc}_\Lambda$ is the $*$-subalgebra of Λ-**equivariant operators** of finite range.

Remark 3. The algebra $\mathcal{A}^{tlc}_\Lambda$ of Λ-equivariant operators of finite range includes many of the operators of interest coming from physics. In particular, many Laplacian operators $\triangle \in \mathcal{B}(\ell^2(\Lambda))$ come from Λ-equivariant operators, and so do Hamiltonians of the form $H = -\triangle + V$, where V is a Λ-equivariant potential.

Definition 6. Let $\mathcal{A}$ be a $*$-algebra. A **trace** on $\mathcal{A}$ is a linear functional $\tau : \mathcal{A} \to \mathbb{C}$ satisfying $\tau(ab) = \tau(ba)$ for any two $a, b \in \mathcal{A}$. The set of all traces of $\mathcal{A}$ forms a $\mathbb{C}$-vector space which we will denote by $\mathrm{Tr}(\mathcal{A})$.

Let $u : \mathbb{R}^d \to \mathbb{R}$ be a smooth bump function with compact support and integral one. We now define a family of maps $w_{\Lambda',u} : \mathcal{A}_\Lambda^{tlc} \to \Delta_\Lambda^0$ parametrized by $\Lambda' \in \Omega_\Lambda$. By duality, these are also maps to Δ_Λ^d. For $A = \pi a \in \mathcal{A}_\Lambda^{tlc}$ and $A_{\Lambda'} = \pi_{\Lambda'} a \in \mathcal{B}(\ell^2(\Lambda'))$ we define the map $w_{\Lambda',u} : A_{\Lambda'} \mapsto f_{A_{\Lambda'}}$ by

$$f_{A_{\Lambda'}}(t) = w_{\Lambda',u}(A)(t) := \sum_{p \in \Lambda'} A_{\Lambda'}(p,p) u(p+t), \tag{10}$$

which is a smooth Λ-equivariant function. As such, it has a cohomology class.

Definition 7. Let $A \in \mathcal{A}_\Lambda^{tlc}$ be a Λ-equivariant operator of finite range. The **cohomology class** $[A_{\Lambda'}]$ **of the operator** $A_{\Lambda'} = \pi_{\Lambda'} a \in \mathcal{B}(\ell^2(\Lambda'))$ is defined to be the cohomology class $[A_{\Lambda'}] = [f_{A_{\Lambda'}}(\star 1)] = [w_{\Lambda',u}(A)(\star 1)] \in H^d(\Omega_\Lambda; \mathbb{R}^d)$.

Lemma 1. *Let $A \in \mathcal{A}_\Lambda^{tlc}$ be a Λ-equivariant operator of finite range. For any two $\Lambda_1, \Lambda_2 \in \Omega_\Lambda$ we have that $[A_{\Lambda_1}] = [A_{\Lambda_2}]$.*

Proof. Let u be a radially symmetric smooth bump function supported in a very small ball (smaller than the inner radius of Λ) and of integral one. Consider the functions $f_i(t) = f_{A_{\Lambda_i}}(t)$ for $i = 1, 2$, where $A_{\Lambda_i} = \pi_{\Lambda_i} A$ and $A \in \mathcal{A}_\Lambda^{tlc}$. Then

$$f_i(t) = \sum_{p \in \Lambda} A_{\Lambda_i}(p,p) u(p+t).$$

Note that since the support of u is small enough, then these functions only depend on the values of the operators A_{Λ_i} along the diagonal, that is, on $A_{\Lambda_i}(p,p)$. For $i = 1, 2$, consider the functions $h_i : \mho_\Lambda \to \mathbb{R}$ defined as follows. For each $p \in \Lambda_i$ we can identify it to a point in $\mho_\Lambda$ by translating Λ_i in such a way that p is translated to $\bar{0}$. Call this map $m_i : \Lambda_i \to \mho_\Lambda$. The map m_i is a bijection onto its image $m_i(\Lambda_i)$, which is dense in $\mho_\Lambda$. Let h_i be the function $h_i : m_i(\Lambda_i) \to \mathbb{R}$ be defined by $h_i(\Lambda_0) = A(m_i^{-1}(\Lambda_0), m_i^{-1}(\Lambda_0))$ for each $\Lambda_0 \in m_i(\Lambda_i)$. This function can be extended to the entire transversal $\mho_\Lambda$ as follows.

Recall that since $A_{\Lambda_i} = \pi_{\Lambda_i} A$ come from a Λ-equivariant operators of finite range $A \in \mathcal{A}_\Lambda^{tlc}$ then there exists an R_A such that if $\Lambda_a, \Lambda_b \in \Omega_\Lambda$ have the property that $\Lambda_a \cap B_{R_A}(0) = \Lambda_b \cap B_{R_A}(0)$ then $A_{\Lambda_a}(p,q) = A_{\Lambda_b}(p,q)$ for any $p, q \in B_{R_A}(0) \cap \Lambda_a$. For any $\Lambda_0 \in m_i(\Lambda_i)$, let $U_{R_A}(p)$ denote the $1/R_A$-neighborhood of Λ_0, that is,

$$U_{R_A}(\Lambda_0) = \{\Lambda' \in \mho_\Lambda : \Lambda' \cap B_{R_A}(0) = \Lambda_0 \cap B_{R_A}(0)\}.$$

As such, for any $\Lambda_a \in m_i(\Lambda_i)$ and any $\Lambda_b \in U_{R_A}(\Lambda_a) \cap m_i(\Lambda_i)$, we have that $h_i(\Lambda_a) = h_i(\Lambda_b)$. Since $m_i(\Lambda_i) \cap U_{R_A}(\Lambda_a)$ is dense in $U_{R_A}(\Lambda_a)$, we can extend h_i to all of $U_{R_A}(\Lambda_a)$ since it is constant on the set $m_i(\Lambda_i) \cap U_{R_A}(\Lambda_a)$, and so we make h_i constant on all of $U_{R_A}(\Lambda_a)$. Since $\mho_\Lambda$ is compact, there are finitely many open sets of the form $U_{R_A}(\Lambda_0)$ for which we need to do this, and thus we can extend h_i to be defined on all of $\mho_\Lambda$. Moreover, it is transversally locally constant.

Comparing h_1 and h_2, we see that if for two $\Lambda_a, \Lambda_b \in \mho_\Lambda$ we have that if $\Lambda_a \in U_{R_A}(\Lambda_b)$, $h_1(\Lambda_a)$ is determined by the value of A determined by the cluster $B_{R_A}(0) \cap \Lambda_a$. Since $h_2(\Lambda_a)$ is also determined by the same clusters and the same Λ-equivariant operator A, $h_2(\Lambda_a) = h_1(\Lambda_b) = h_1(\Lambda_a)$. Since this argument works for any clopen neighborhood of the form $U_{R_A}(\Lambda_0)$ and $\mho_\Lambda$ is compact, $h_1(\Lambda_0) = h_2(\Lambda_0)$ for any $\Lambda_0 \in \mho_\Lambda$. So the functions h_1 and h_2 are the same transversally locally constant functions, and f_1, f_2 are obtained by smoothing h_1, h_2 along the leaf direction with u. As such, they belong to the same cohomology class. $\square$

Remark 4. Given Lemma 1 we will supress the subscript Λ' from the operators $A_{\Lambda'}$ because the cohomology class of their traces will be independent of representative.

By Lemma 1, the map $w_{\Lambda',u}$ does not depend on Λ' and thus we get a map $w_u : \mathcal{A}_\Lambda^{tlc} \to H^d(\Omega_\Lambda; \mathbb{R})$.

Remark 5. Note that to construct a Λ-equivariant function $w_u(A)(\star 1)$ from $A \in \mathcal{A}_\Lambda^{tlc}$, and therefore to assign it a cohomology class, we used a smooth compactly supported function u. However, in the definition of the cohomology class of A there is no reference to u. We will see in Proposition 1 that for the purposes needed, this class is independent of which u we take, as long as it is smooth, compactly supported, and has integral one.

Remark 6. Note that we have defined the cohomology class of operators $A \in \mathcal{A}_\Lambda^{tlc}$, a dense subalgebra of $\mathcal{A}_\Lambda^{fin}$, which is itself a dense subalgebra of $\mathcal{A}_\Lambda$. This definition cannot be extended to the subalgebra $\mathcal{A}_\Lambda^{fin}$ or full C*-algebra $\mathcal{A}_\Lambda$: the function $f_A(t)$ in (10) is pattern equivariant in the sense of Definition 1. In the language of [Kel03], it is *strongly* pattern equivariant. Elements in the subalgebra $\mathcal{A}_\Lambda^{fin}$ (or the closure $\mathcal{A}_\Lambda$) may no longer give *strongly* pattern equivariant functions, but *weakly* pattern equivariant functions (in the language of [Kel03]). Therefore the cohomology in definition 2, the *strongly* pattern-equivariant cohomology of [Kel03], does not describe the cohomology class of $f_A(t)$ for $A \in \mathcal{A}_\Lambda$. However, the *weakly* pattern-equivariant cohomology may describe the cohomology class of $f_A(t)$ in that case. Thus, the traces which define the function $f_A(t)$ may not extend to the subalgebra $\mathcal{A}_\Lambda^{fin}$ or the closure $\mathcal{A}_\Lambda$ of $\mathcal{A}_\Lambda^{fin}$. This is why we work with operators in the dense subalgebra $\mathcal{A}_\Lambda^{tlc}$. By Remark 3, it contains plenty of interesting operators.

For any $A \in \mathcal{A}_\Lambda^{fin}$ and any bounded set $B \subset \mathbb{R}^d$, we denote by $A_{\Lambda'}|_B$ the restriction of $A_{\Lambda'}$ to the finite dimensional subspace $\ell^2(\Lambda' \cap B)$ of $\ell^2(\Lambda')$.

5. Traces and asymptotic cycles

We now recall the relevant ergodic theoretic results from [ST15]. Let Λ be an RFT Delone set and recall the definition of the rapidly expanding subspace $E_\Lambda^+ \subset H^d(\Omega_\Lambda; \mathbb{R}^d)$. For a RFT Delone set there exists an expansive matrix $A \in GL^+(d, \mathbb{R})$ satisfying the conjugacy equation (5). For any set $B_0 \subset \mathbb{R}^d$ with non-empty interior, containing the origin and with a regular boundary (a Lipschitz domain), we define a one-parameter family of sets for $T > 1$:

$$B_T = exp\left(d\frac{a\log T}{|A|}\right) B_0. \tag{11}$$

Here $a \in \mathfrak{gl}(d, \mathbb{R})$ satisfies $A = exp(a)$. These sets have the property that $\mathrm{Vol}(B_T) = \mathrm{Vol}(B_0) \cdot T^d$.

Using the basis in (6), for a Lipschitz domain B_0 containing the origin and $\Lambda_0 \in \Omega_\Lambda$, for any $f \in C_{tlc}^\infty(\Omega_\Lambda)$ we can write its ergodic integral as

$$\int_{B_T} f \circ \varphi_t(\Lambda_0)\, dt = \sum_{(i,j,k)\in I_\Lambda^+} \alpha_{i,j,k}(f) \int_{B_T} \eta_{i,j,k} + \mathcal{O}(|\partial B_T|), \tag{12}$$

where the $\eta_{i,j,k}$ are the representatives of the basis in (6). Moreover, by [ST15, Proposition 6],

$$\left|\alpha_{i,j,k}(f)\int_{B_T}\eta_{i,j,k}\right| \leq C_{\Lambda,B_0}|\alpha_{i,j,k}(f)|L(i,j,T)T^{d\frac{\log|\nu_i|}{\log\nu_1}}, \tag{13}$$

where $L(i,j,T)$ is a non-negative power of $\log T$ (see (15) and (16) below), for all $T > 0$. The main result of [ST15] states that there exist $|I_\Lambda^+|$ closed, $\mathbb{R}^d$-invariant, Λ-equivariant currents $\{\mathfrak{C}_{i,j,k}\}_{(i,j,k)\in I_\Lambda^+}$ which control the growth of ergodic integrals (12).

We now recall the construction of the asymptotic cycles from [ST15, §5.3]. For $T > 3$ and an index $(i,j,k) \in I_\Lambda^+$, define the Λ-equivariant currents $\mathfrak{C}_{i,j,k}^{B_0,T}$ as

$$\begin{aligned}\mathfrak{C}_{i,j,k}^{B_0,T} : \eta \mapsto \mathfrak{C}_{i,j,k}^{B_0,T}(\eta) &= \int_{B_T}\eta - \sum_{\substack{(i',j',k')\leq(i,j,k)\\ k'\neq k}} \alpha_{i',j',k'}(\eta)\int_{B_T}\eta_{i',j',k'} \\ &= \sum_{\substack{(i',j',k')\geq(i,j,k)\\ k'\neq k}} \alpha_{i',j',k'}(\eta)\int_{B_T}\eta_{i',j',k'} + \mathcal{O}(|\partial B_T|)\end{aligned} \tag{14}$$

for any $\eta \in \Delta_\Lambda^d$. Let $s_i = d\frac{\log|\nu_i|}{\log|\nu_1|}$. We can now average to define the currents. For an index $(i,j,k) \in I_\Lambda^{+,>}$,

$$\mathfrak{C}_{i,j,k}([\eta]) = \limsup_{T\to\infty} \frac{1}{(\log T)^{j-1}T^{s_i}}\mathfrak{C}_{i,j,k}^{B_0,T}(\eta), \tag{15}$$

which, by (13), exists. In this case $L(i,j,T) = (\log T)^{j-1}$. For an index $(i,j,k) \in I_\Lambda^{+,=}$,

$$\mathfrak{C}_{i,j,k}([\eta]) = \limsup_{T\to\infty} \frac{1}{(\log T)^{j}T^{s_i}}\mathfrak{C}_{i,j,k}^{B_0,T}(\eta). \tag{16}$$

In this case $L(i,j,T) = (\log T)^j$.

In the notation above, we emphasize that the functionals $\mathfrak{C}_{i,j,k}$ only depend on the cohomology class $[f]$ of f, which makes them cycles. These functionals yield asymptotic cycles $\mathfrak{C}_{i,j,k}$ in the sense that they are defined by an averaging procedure along orbits of the $\mathbb{R}^d$ action. In fact, $\mathfrak{C}_{1,1,1}$ is the Schwartzman-Ruelle-Sullivan asymptotic cycle corresponding to the leading eigenvalue $\nu_1 = |A|$. Close examination will convince the reader that they also satisfy $\mathfrak{C}_{i,j,k}([f]) = 0$ if and only if $\alpha_{i,j,k}(f) = 0$. In fact, $\mathfrak{C}_{i,j,k}$ is a non-zero multiple of $\alpha_{i,j,k}$. By scaling the forms $\eta_{i,j,k}$ appropriately, that is, by scaling $\eta_{i,j,k}$ such that

$$\limsup_{T\to\infty} \frac{1}{L(i,j,k)T^{s_i}}\int_{B_T}\eta_{i,j,k} = 1, \tag{17}$$

we can assume that indeed $\mathfrak{C}_{i,j,k} = \alpha_{i,j,k}$.

For $(i,j,k) \in I_\Lambda^+$, define the map $\tau_{i,j,k} : \mathcal{A}_\Lambda^{tlc} \to \mathbb{R}$ by

$$\tau_{i,j,k} : A \mapsto \mathfrak{C}_{i,j,k}([A]) = \mathfrak{C}_{i,j,k}([w_{\Lambda,u}(A)(\star 1)]). \tag{18}$$

Proposition 1. *The maps $\tau_{i,j,k}$ defined in (18) are traces on $\mathcal{A}_\Lambda^{tlc}$.*

Proof. Let $u : \mathbb{R}^d \to \mathbb{R}$ be a smooth function with compact support and integral one. We now turn to applying the currents $\mathfrak{C}_{i,j,k}$ to the forms $w_{\Lambda,u}(A)(\star 1)$ obtained through the map (10) for operators $A \in \mathcal{A}_\Lambda^{tlc}$. It suffices to show that $\tau_{i,j,k}$ is positive and that $\tau_{i,j,k}(AA^*) = \tau_{i,j,k}(A^*A)$ for $A = \{A_{\Lambda'}\}_{\Lambda'\in\Omega_\Lambda}$. Let $A \in \mathcal{A}_\Lambda^{tlc}$.

We first compute the images of AA^* and A^*A, respectively, under the map $w_{\Lambda,u}$. Let $a \in \mathcal{K}_\Lambda^{tlc}$ satisfy $\pi_\Lambda a = A$. Recalling the convolution product and the $*$-involution for kernels of finite range, and (10),

$$
\begin{aligned}
f_{AA^*}(t) &= w_{\Lambda,u}(AA^*)(t) \\
&= \sum_{p\in\Lambda}(AA^*)_\Lambda(p,p)u(p+t) = \sum_{p\in\Lambda}\sum_{x\in\Lambda} a(p,\Lambda,x)a^*(x,\Lambda,p)u(p+t) \\
&= \sum_{p\in\Lambda}\sum_{x\in\Lambda} a(p,\Lambda,x)\bar{a}(p,\Lambda,x)u(p+t) = \sum_{p\in\Lambda}\sum_{x\in\Lambda} |a(p,\Lambda,x)|^2 u(p+t).
\end{aligned} \tag{19}
$$

Likewise:

$$
\begin{aligned}
f_{A^*A}(t) &= w_{\Lambda,u}(A^*A)(t) \\
&= \sum_{p\in\Lambda}(A^*A)_\Lambda(p,p)u(p+t) = \sum_{p\in\Lambda}\sum_{x\in\Lambda} a^*(p,\Lambda,x)a(x,\Lambda,p)u(p+t) \\
&= \sum_{p\in\Lambda}\sum_{x\in\Lambda} \bar{a}(x,\Lambda,p)a(x,\Lambda,p)u(p+t) = \sum_{p\in\Lambda}\sum_{x\in\Lambda} |a(x,\Lambda,p)|^2 u(p+t).
\end{aligned} \tag{20}
$$

Note that both f_{AA*} and f_{A*A} are positive. Therefore, the difference is

$$
f_{AA^*}(t) - f_{A^*A}(t) = \sum_{p\in\Lambda}\left(\sum_{q\in\Lambda}|a(p,\Lambda,q)|^2 - |a(q,\Lambda,p)|^2\right)u(p+t) = f_{AA^*-A^*A}(t).
$$

Let

$$
D_T = \int_{B_T} f_{AA^*-A^*A}(t)\,dt
$$

and R_a be the range of $a \in \mathcal{K}_\Lambda^{tlc}$. By definition, for any $p,q\in\Lambda$ with $|p-q|\geq R_a$, we have that $a(p,\Lambda,q)=0$. Denote by $r_u>0$ a number such that the support of u is contained in the ball of radius r_u around the origin. Denote by $B_T^{A,r_u}\subset B_T$ the subset

$$
B_T^{A,r_u} = \{x\in B_T : \operatorname{dist}(x,\partial B_T) > 2(r_u+R_a)\},
$$

which is not empty for all large enough $T>0$. For $r>0$ and a subset $A\subset\mathbb{R}^d$ let

$$
\partial_r A = \{x\in\mathbb{R}^d : \operatorname{dist}(x,\partial A)\leq r\}
$$

be the r-neighborhood of ∂B_T.

Suppose $p_1\in\Lambda$ is such that $p_1 + B_{r_u}(0)\subset B_T^{A,r_u}$. Since the integral of u is 1, then p_1 contributes $\sum_{q\in\Lambda}|a(p_1,\Lambda,q)|^2 - |a(q,\Lambda,p_1)|^2$ in the sum under the integral D_T. If q_1 is such that $a(p_1,\Lambda,q_1)\neq 0$ or $a(q_1,\Lambda,p_1)\neq 0$, then since $|p_1-q_1|\leq R_a$, $p_1\in B_T^{A,r_u}$ and the integral of u is one, q_1 contributes $\sum_{z\in\Lambda}|a(q_1,\Lambda,z)|^2 - |a(z,\Lambda,q_1)|^2$ in the sum under the integral D_T. However, since $a(p_1,\Lambda,q_1)\neq 0$ or $a(q_1,\Lambda,p_1)\neq 0$, the contribution cancels out that one which came from its interaction to q_1. Since this sort of cancellation happens for all pairs of points $p^*, q^*\in B_T^{A,r_u}$

with $a(p^*, \Lambda, q^*) \neq 0$ or $a(p^*, \Lambda, q^*) \neq 0$,

$$\begin{aligned}
\left|\int_{B_T} f_{AA^*-A^*A}(t)\,dt\right| &\leq \left|\int_{B_T\setminus B_T^{A,r_u}} f_{AA^*-A^*A}(t)\,dt\right| \leq \int_{\partial_{2(r_u+R_a)}B_T} |f_{AA^*-A^*A}(t)|\,dt \\
&\leq \#(\partial_{2(r_u+R_a)}B_T\cap\Lambda)\|A\|^2 \\
&\leq 4(r_u+R_a)D_\Lambda\|A\|^2\mathrm{Vol}(B_0)T^{d\left(1-\frac{\log\lambda_d}{\log\nu_1}\right)} = \mathcal{O}(|\partial B_T|)
\end{aligned} \tag{21}$$

for all T large enough, where we have used [ST15, Lemma 5] in the last inequality and D_Λ only depends on the Delone set Λ. Comparing (21) with its expansion through (12), we have that $\alpha_{i,j,k}(f_{AA^*-A^*A}) = 0$ for all $(i,j,k)\in I_\Lambda^+$ and thus $\mathfrak{C}_{i,j,k}([AA^*-A^*A]) = 0$ for all $(i,j,k)\in I_\Lambda^+$. As such, $\tau_{i,j,k}(AA^*) = \mathfrak{C}_{i,j,k}([AA^*]) = \mathfrak{C}_{i,j,k}([A^*A]) = \tau_{i,j,k}(A^*A)$. The proposition follows since this works for any bump function u. □

Let $\tau_{i,j,k} = \mathfrak{C}_{i,j,k}\circ w : \mathcal{A}_\Lambda^{tlc}\to\mathbb{C}$ be the traces above.

Corollary 1. *The space*

$$\mathrm{Tr}_\Lambda^+(\mathcal{A}_\Lambda^{tlc}) := \bigoplus_{(i,j,k)\in I_\Lambda^+} \mathrm{span}\,\tau_{i,j,k}$$

is a subspace of dimension $\dim E_\Lambda^+$ *of the space of traces* $\mathrm{Tr}(\mathcal{A}_\Lambda^{tlc})$.

Proof of Theorem 2. Let B_0 be a Lipschitz domain, $\mathcal{A}\in\mathcal{A}_\Lambda^{tlc}$, and u a smooth bump function of compact support and integral one. Using the map in (10), let $f_A(t) = w_{\Lambda,u}(A)(t)$ be a Λ-equivariant function and denote

$$I_T = \int_{B_T} f_A(t)\,dt.$$

We want to bound the quantity

$$\begin{aligned}
\left|\sum_{q\in\Lambda\cap B_T} A(q,q) - I_T\right| &= \left|\sum_{q\in\Lambda\cap B_T} A(q,q) - \int_{B_T} f_A(t)\,dt\right| \\
&= \left|\sum_{p\in\Lambda\cap B_T} A(p,p) - \int_{B_T}\sum_{p\in\Lambda} A(p,p)u(p+t)\,dt\right|.
\end{aligned}$$

Let $r_u > 0$ be such that $\mathrm{supp}(u)\subset B_{r_u}(0)$ and let T be large enough so that $B_{r_u}(0)\subset B_T$. Suppose $p\in\Lambda$ is such that $p+B_{r_u}(0)\subset B_T$. Since the integral of u is 1, then p contributes $A(p,p)$ in the sum under the integral I_T, and so it cancels with the same quantity in $\sum_{p\in\Lambda}A(p,p)$. This happens to all $p\in\Lambda\cap B_T$ with the exception of those $q\in\Lambda\cap B_T$ with distance to ∂B_T less than or equal to $2r_u$. Thus

$$\begin{aligned}
\left|\sum_{q\in\Lambda\cap B_T} A(q,q) - I_T\right| &= \left|\sum_{p\in\Lambda\cap B_T} A(p,p) - \int_{B_T}\sum_{p\in\Lambda} A(p,p)u(p+t)\,dt\right| \\
&\leq \int_{\partial_{2r_u}B_T}\left|\sum_{p\in\Lambda} A(p,p)u(p+t)\right|\,dt \\
&\leq \#(\partial_{2r_u}B_T\cap\Lambda)\|A\| \\
&\leq 4r_uD_\Lambda\|A\|\mathrm{Vol}(B_0)T^{d\left(1-\frac{\log\lambda_d}{\log\nu_1}\right)} = \mathcal{O}(|\partial B_T|).
\end{aligned} \tag{22}$$

Therefore, up to terms of order $|\partial B_T|$, $\sum_{q\in\Lambda\cap B_T} A(q,q)$ and $\int_{B_T} f_A(t)\,dt$ agree. For every index $(i,j,k)\in I_\Lambda^+$ define the function

$$\Psi_{i,j,k}^{B_0}(T) := \frac{1}{L(i,j,T)T^{\frac{\log|\nu_i|}{\log\nu_1}}}\int_{B_T}\eta_{i,j,k} : \mathbb{R}^+\to\mathbb{R}. \tag{23}$$

By (17), these functions satisfy $\limsup_{T\to\infty}\Psi_{i,j,k}^{B_0}(T)=1$. Using (14),

(24)

$$\begin{aligned}
\mathfrak{C}_{i,j,k}^{B_0,T}(f_A(\star 1)) &= \int_{B_T} f_A(t)\,dt - \sum_{\substack{(i',j',k')\le(i,j,k)\\ k'\neq k}} \alpha_{i',j',k'}([f_A])\int_{B_T}\eta_{i',j',k'} \\
&= \int_{B_T} f_A(t)\,dt - \sum_{\substack{(i',j',k')\le(i,j,k)\\ k'\neq k}} \mathfrak{C}_{i',j',k'}([f_A])\Psi_{i',j',k'}^{B_0}(T)L(i',j',T)T^{\frac{\log|\nu_{i'}|}{\log\nu_1}} \\
&= \int_{B_T} f_A(t)\,dt - \sum_{\substack{(i',j',k')\le(i,j,k)\\ k'\neq k}} \tau_{i',j',k'}(A)\Psi_{i',j',k'}^{B_0}(T)L(i',j',T)T^{\frac{\log|\nu_{i'}|}{\log\nu_1}} \\
&= \mathrm{tr}(A|_{B_T}) - \sum_{\substack{(i',j',k')\le(i,j,k)\\ k'\neq k}} \tau_{i',j',k'}(A)\Psi_{i',j',k'}^{B_0}(T)L(i',j',T)T^{\frac{\log|\nu_{i'}|}{\log\nu_1}} + \mathcal{O}(|\partial B_T|).
\end{aligned}$$

As such, (3) is obtained through (15), (16) and (24). □

Let $\mathcal{S}_\Lambda\subset\mathcal{A}_\Lambda^{tlc}$ be the subset of self-adjoint elements of $\mathcal{A}_\Lambda^{tlc}$. That is, for $A\in\mathcal{S}_\Lambda$, $A_{\Lambda'}\in\mathcal{B}(\ell^2(\Lambda'))$ is self-adjoint and Λ-equivariant for every $\Lambda'\in\Omega_\Lambda$. Recall that for any any self-adjoint operator $A\in\mathcal{B}(\ell^2(\Lambda))$ we can construct the C*-algebra $C(A)=C(A,1)$ which is generated by A and the identity 1. That is, it is the completion of $P(A)$ in the operator norm of the set of all polynomials in A. The continuous functional calculus states that this algebra is $*$-isomorphic to $C(\sigma(A))$.

Proposition 2. *Let $A\in\mathcal{S}_\Lambda$ be a family of self-adjoint operators and denote by $A_{\Lambda'}$ the associated self-adjoint operator in $\mathcal{B}(\ell^2(\Lambda'))$ for any $\Lambda'\in\Omega_\Lambda$. Then for any $\Lambda'\in\Omega_\Lambda$ there is an injective map $\Theta_{A_{\Lambda'}}:\mathrm{Tr}_\Lambda^+(\mathcal{A}_\Lambda^{tlc})\to\mathrm{Tr}(C(A_{\Lambda'},1))$.*

Proof. Let $\varphi(A)\in P(A)$ be a polynomial in A. Then $\varphi(A)$ is a Λ-equivariant operator of finite range and $\tau_{i,j,k}(\varphi(A))$ is well defined for any $\tau_{i,j,k}\in\mathrm{Tr}_\Lambda^+(\mathcal{A}_\Lambda^{tlc})$. Let $\varphi\in C([-\|A\|-2,\|A\|+2])$ be a continuous function and denote by $\{\varphi_n\}$ a Cauchy sequence of polynomials which converge to φ uniformly in $C([-\|A\|-2,\|A\|+2])$ under the supremum norm. By the continuous functional calculus, $\{\tau_{i,j,k}(\varphi_n(A))\}_n$ is a Cauchy sequence, so $\tau_{i,j,k}(\varphi(A))=\lim_{n\to\infty}\tau_{i,j,k}(\varphi_n(A))$ is the extension to $\mathrm{Tr}(C(A_{\Lambda'},1))$ of $\mathrm{Tr}_\Lambda^+(\mathcal{A}_\Lambda^{tlc})$. □

Given a self-adjoint operator $A\in\mathcal{S}_\Lambda$ let $J_A\subset\mathbb{R}$ be a closed interval of finite length which contains the spectrum $\sigma(A)$ of A and let $\Lambda'\in\Omega_\Lambda$. For each index $(i,j,k)\in I_\Lambda^+$, by Proposition 2, there exists a unique regular countably additive Borel measure $\rho_{i,j,k}^A$ defined by

$$\rho_{i,j,k}^A(\varphi)=\Theta_{\Lambda'}(\tau_{i,j,k})(\varphi(A))$$

for any $\varphi \in C(J_A)$. In a slight abuse of notation, we will sometimes denote $\rho^A_{i,j,k}(\varphi) = \tau_{i,j,k}(\varphi(A))$ even though we implicitly use the map $\Theta_{\Lambda'}$ to extend the traces.

Proof of Theorem 1. For a RFT Delone set $\Lambda' \in \Omega_\Lambda$ and a Lipschitz domain B_0, let $A \in \mathcal{B}(\ell^2(\Lambda'))$ be defined by a self-adjoint $A \in \mathcal{S}_\Lambda$, and let $\varphi \in C(J_A)$ be a polynomial. We take the functions $\Psi^{B_0}_{i,j,k}$ to be the same ones as in (23). In [LS03, Theorem 4.7], it is shown that (see the end of the proof of Theorem 4.7)

$$|\mathrm{tr}(\varphi(A|_{B_T})) - \mathrm{tr}(\varphi(A)|_{B_T})| \le C|\partial_{N\cdot R_a} B_T|, \tag{25}$$

where N is the degree of φ, and R_a denotes the range of the kernel corresponding to A. We note that the term on the right hand side of (25) is $\mathcal{O}(|\partial B_T|)$, since we have fixed φ. Thus, for $T > 0$,

(26)

$$\begin{aligned}
&\mathrm{tr}(\varphi(A|_{B_T})) - \sum_{\substack{(i',j',k')\le(i,j,k)\\ k'\neq k}} \rho^A_{i',j',k'}(\varphi)\Psi^{B_0}_{i',j',k'}(T)L(i',j',T)T^{\frac{\log|\nu_{i'}|}{\log\nu_1}} \\
&\quad= \mathrm{tr}(\varphi(A)|_{B_T}) - \sum_{\substack{(i',j',k')\le(i,j,k)\\ k'\neq k}} \tau_{i',j',k'}(\varphi(A))\Psi^{B_0}_{i',j',k'}(T)L(i',j',T)T^{\frac{\log|\nu_{i'}|}{\log\nu_1}} + \mathcal{O}(|\partial B_T|) \\
&\quad= \mathrm{tr}(\varphi(A)|_{B_T}) - \sum_{\substack{(i',j',k')\le(i,j,k)\\ k'\neq k}} \mathfrak{C}_{i',j',k'}([f_{\varphi(A)}])\Psi^{B_0}_{i',j',k'}(T)L(i',j',T)T^{\frac{\log|\nu_{i'}|}{\log\nu_1}} + \mathcal{O}(|\partial B_T|) \\
&\quad= \mathrm{tr}(\varphi(A)|_{B_T}) - \sum_{\substack{(i',j',k')\le(i,j,k)\\ k'\neq k}} \alpha_{i',j',k'}([f_{\varphi(A)}])\Psi^{B_0}_{i',j',k'}(T)L(i',j',T)T^{\frac{\log|\nu_{i'}|}{\log\nu_1}} + \mathcal{O}(|\partial B_T|) \\
&\quad= \mathfrak{C}^{B_0,T}_{i,j,k}([f_{\varphi(A)}]) + \mathcal{O}(|\partial B_T|).
\end{aligned}$$

Therefore,

(27)

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{L(i,j,T)T^{d\frac{\log|\nu_i|}{\log\nu_1}}}\left(\mathrm{tr}(\varphi(A|_{B_T})) - \sum_{\substack{(i',j',k')\le(i,j,k)\\ k'\neq k}} \rho^A_{i',j',k'}(\varphi)\Psi^{B_0}_{i',j',k'}(T)L(i',j',T)T^{\frac{\log|\nu_{i'}|}{\log\nu_1}}\right) \\
&\quad= \limsup_{T\to\infty} \frac{\mathfrak{C}^{B_0,T}_{i,j,k}([f_{\varphi(A)}(\star 1)])}{L(i,j,T)T^{d\frac{\log|\nu_i|}{\log\nu_1}}} \\
&\quad= \tau_{i,j,k}(\varphi(A)) = \rho^A_{i,j,k}(\varphi),
\end{aligned}$$

which proves Theorem 1. □

Remark 7. One can see from the proof of Theorem 1 that the difficulty in extending Theorem 1 to all continuous functions lies in the bound

$$\mathrm{tr}(\varphi(A|_{B_T})) - \mathrm{tr}(\varphi(A)|_{B_T}) \le C|\partial_{N\cdot R_a} B_T| \tag{28}$$

and in particular, the dependence on the degree of φ on the right hand side.

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